

Algebraic Curves: Projects

0 Projects

Your project should be a short (3-10 page¹) essay exploring some part of the module in depth. It should form a coherent narrative (e.g. written in full sentences). You can give references to material from the lecture notes or to material you find in books or online (and you *should* give references to anything you actually use to help you: references are not part of the page-count). Don't bother to include an abstract.

There are three suggested projects below. The first is quite exploratory (see how much you can prove using the tools we already developed). The other two are basically guided exercises: sequences of questions that should help you explore the suggested topic. **How-ever** you should not write your project as a sequence of answers (like "Question 1. Show that a snoboidal curve can have at most 15 gloom points. Answer. ..."). Instead, having acquired an understanding of snoboidal curves² for yourself by doing the guided exercise, explain it in your project as if you were giving a lecture to someone else (someone who hadn't seen the guided exercise but who had attended the whole of MATH323/423).

Of course, if you prefer, you can ignore the guided exercises and just go and read about the material yourself in a book. If you have independent ideas for a project, let me know and I can advise.

0.1 Grading

The project will be awarded a letter grade according to the university's marking guidelines, as outlined in the Part II Handbook. I have modified these slightly for this specific project (which is more about trying to work stuff out for yourself than reading "a comprehensive range of materials").

- A Grades: Exemplary range and depth of attainment of intended learning outcomes, secured by discriminating command of material and techniques from the course, and by deployment of considered judgement relating to key issues, concepts and procedures.
- B Grades: Conclusive attainment of virtually all intended learning outcomes, clearly grounded on a close familiarity with material and techniques from the course, constructively utilised to reveal appreciable depth of understanding.

¹Projects falling outside of this page limit will be penalised.

²or resultants or whatever.

- C Grades: Clear attainment of most of the intended learning outcomes, some more securely grasped than others, resting on a circumscribed range of material and techniques from the course, and displaying a variable depth of understanding.
- D Grades: Acceptable attainment of intended learning outcomes, displaying a qualified familiarity with a minimally sufficient range of material and techniques from the course, and a grasp of the analytical issues and concepts which is generally reasonable, albeit insecure.
- Marginal Fail: Attainment deficient in respect of specific intended learning outcomes, with mixed evidence as to the depth of knowledge and weak deployment of arguments or deficient manipulations.
- Fail: Attainment of intended learning outcomes appreciably deficient in critical respects, lacking secure basis in relevant factual and analytical dimensions.

For these general statements to make sense, I have to tell you what the “intended learning outcomes” for this project are.

0.2 Intended learning outcomes

- Read about or actively explore an idea or circle of ideas related to the algebraic curves course but going beyond the main sequence of lectures.
- Construct a coherent, concise and correct mathematical narrative which explains these ideas to your assessor and your peers. You should aim to be comprehensible to someone who has taken the same course, but may have focused on a different project.
- Demonstrate judgement about the importance and relevance of material you include.
- Where appropriate, illustrate³ your explanation with carefully-chosen examples.
- Where appropriate, extract and establish general patterns from specific worked examples.
- Where appropriate, demonstrate an ability to unpack and interrogate proofs rather than simply copying them out or paraphrasing them. This means:
 - Extracting and summarising the key new idea or construction at the heart of a proof.
 - Reworking proofs with extra simplifying assumptions to allow the key idea to shine through.
 - Identifying points in a proof that need amplifying in more detail for you to understand them.
 - Correctly evaluating which parts of a proof are easy or routine and which require more thought or new ideas.

³This doesn't have to mean literally a picture!

1 Constraints on singularities

In lectures, we established some rudimentary bounds on the kinds of singularities that can appear for irreducible cubics and quartics. This project is to extend these results as much as you can.

1. How many double points can an irreducible quintic curve have? Can they all lie on a conic? What about triple points, quadruple points, or combinations of points with different multiplicities?
2. What about sextics? Septics? What is the most general result you can prove that is valid for all d ?
3. Can you find curves which realise these bounds? For example, a quartic with three double points? Or a quintic with whatever singularities it is allowed? You could try looking online or in a book, or you could use our theorem on existence of curves with constraints to find them for yourself.

2 Cayley-Bacharach theorem

The Cayley-Bacharach theorem says:

Theorem 2.1. *If $C = \{f = 0\}$, $C' = \{g = 0\}$ and $C'' = \{h = 0\}$ are cubic curves such that*

$$C \cap C' = \{p_1, \dots, p_9\} \text{ and } C \cap C'' = \{p_1, \dots, p_8, p'_9\}$$

then $p_9 = p'_9$.

Given a collection of points p_1, \dots, p_n , we will write $\Sigma(p_1, \dots, p_n)$ for the space of cubic polynomials that vanish at all these points. Recall that the space of *all* cubics is 10-dimensional, and each point constraint cuts down the dimension by at most one. Therefore $\Sigma(p_1, \dots, p_8)$ is at least 2-dimensional.

Proposition 2.2. *If no four of p_1, \dots, p_8 lie on a line and no seven lie on a conic then $\Sigma(p_1, \dots, p_8)$ is precisely 2-dimensional.*

1. Assume Proposition 2.2. By thinking about the subspace spanned by $f, g, h \in \Sigma(p_1, \dots, p_8)$, show that h must vanish on the whole of $C \cap C'$ (in particular $p_9 \in C''$). (Hint: Can you express h in terms of f and g ?)

What follows is a guided proof of Proposition 2.2.

2. Suppose six of the points, say p_1, \dots, p_6 , lie on a conic $D = \{\delta = 0\}$ and let $E = \{\epsilon = 0\}$ be the line through the remaining two points p_7, p_8 . Let $q_9 \in D$ be a seventh point on the conic. Show that any cubic $\sigma \in \Sigma(p_1, \dots, q_9)$ is a multiple of $\delta\epsilon$ so that $\dim \Sigma(p_1, \dots, p_8, q_9) = 1$. (Hint: Apply Bézout's theorem to $D \cap \{\sigma = 0\}$.) Deduce that $\dim \Sigma(p_1, \dots, p_8) \leq 2$.
3. Suppose no six points lie on a conic, but three (say p_1, p_2, p_3) lie on a line $E = \{\epsilon = 0\}$. Let $q_9 \in E$ be a fourth point on this line. Show that any cubic $\sigma \in \text{cubic}(p_1, \dots, q_9)$ factorises as $\delta\epsilon$ for some quadratic δ which vanishes on the five points p_4, \dots, p_8 . Show that there is a unique such δ up to scale, so again $\dim \Sigma(p_1, \dots, p_8, q_9) = 1$ and $\dim \Sigma(p_1, \dots, p_8) \leq 2$.
4. Finally, suppose no six points lie on a conic and no three lie on a line. Let $E = \{\epsilon = 0\}$ be the line through p_1 and p_2 and let q_9, q_{10} be two further points on E . Show that the only cubic which vanishes on $p_1, \dots, p_8, q_9, q_{10}$ is zero, and deduce that $\dim \Sigma(p_1, \dots, p_8) \leq 2$.
5. A beautiful corollary of the Cayley-Bacharach theorem is *Pascal's mystic hexagon*: let p_1, \dots, p_6 be six points on a conic D (hexagon inscribed in D) and let $\ell_1, m_1, \ell_2, m_2, \ell_3, m_3$ be the (extended) sides (in opposite pairs). Let p_7, p_8, p_9 be the intersection points $p_7 = \ell_1 \cap m_1, p_8 = \ell_2 \cap m_2, p_9 = \ell_3 \cap m_3$. Then p_7, p_8, p_9 are collinear. Deduce this from the Cayley-Bacharach theorem (Hint: Let L be the line through p_7 and p_8 and take $C = \ell_1 \cup \ell_2 \cup \ell_3, C' = m_1 \cup m_2 \cup m_3$ and $C'' = D \cup L$.)

3 Resultants

Fix a field K and let $f, g \in K[z]$ be two polynomials

$$\begin{aligned}f(z) &= a_0 + a_1z + \cdots + a_mz^m \\g(z) &= b_0 + a_1z + \cdots + b_nz^n\end{aligned}$$

with $a_i, b_j \in K$.

Let $Q_{\leq D}$ denote the space of polynomials of degree at most D in z . Consider the linear map $\phi: Q_{\leq n-1} \times Q_{\leq m-1} \rightarrow Q_{\leq m+n-1}$ defined by $\phi(\alpha, \beta) = \alpha f + \beta g$.

1. If we take the basis $1, z, \dots, z^D$ of $Q_{\leq D}$, this linear map can be written as an $(m+n)$ -by- $(m+n)$ matrix. Find this matrix.

The determinant of this matrix is called the *resultant* of f and g , written $\text{res}(f, g)$. It is an element of the field K .

2. Show that $\text{res}(f, g) = 0$ if and only if f and g have a common factor (if you get stuck, see Walker, Chapter I, §9).
3. Deduce that there exist polynomials a of degree $< n$ and b of degree $< m$ such that $\text{res}(f, g) = af + bg$.
4. Calculate some simple resultants, for example

$$\text{res}(z^3 - 3z^2 + 2z, z^2 - z - 2).$$

(Can you explain why this specific resultant vanishes?)

If $g = df/dz$ then the resultant $\text{res}(f, df/dz)$ has a special name: it is called the *discriminant*.

5. Explain why the discriminant of f vanishes if and only if f has a repeated root.
6. Calculate the discriminant of a quadratic $f(z) = az^2 + bz + c$ and the cubic $f(z) = z^3 + az + b$.

We will now use resultants can to find intersections between curves. Given a polynomial $f(x, y) \in k[x, y]$, we can think of it as a polynomial $\tilde{f}(y) \in K[y]$ with coefficients in $K = k(x)$, the field of rational functions of x . Given $f, g \in k[x, y]$, the resultant $\text{res}(\tilde{f}, \tilde{g}) \in K = k(x)$ is a rational function (actually a polynomial) in x . We know that $\text{res}(\tilde{f}, \tilde{g}) = a\tilde{f} + b\tilde{g}$ for some $a, b \in K[y]$, and since $C \cap C' = \{\tilde{f} = \tilde{g} = 0\}$ this implies that $\text{res}(\tilde{f}, \tilde{g})$ vanishes on $C \cap C'$.

7. Calculate $\text{res}(\tilde{f}, \tilde{g})$ when

$$- f(x, y) = x^2 + y^2 - 1 \text{ and } g(x, y) = y.$$

$$- f(x, y) = 2x^2 + y^2 - 1 \text{ and } g(x, y) = x^2 + 2y^2 - 1.$$

Use this to find the x -coordinates of the intersection points between $\{f = 0\}$ and $\{g = 0\}$ in each case. Check that this is right by finding the intersections directly!

8. Consider the curves $\{2x^2 + y^2 = 1\}$ and $\{x^3 + y^3 = 1\}$. The intersection point $(0, 1)$ is easy to spot. How many other intersection points are there (over \mathbb{C})?