Algebraic Curves: Worksheet 1

Classifying Complex Conics

Let $P(x, y)$ be a quadratic polynomial, i.e. one of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

with at least one of a, b, c non-zero. The curve $P = 0$ is called a *conic*. This includes ellipses (in particular circles), hyperbolae, and parabolae, as well as "degenerate" conics like $xy = 0$ or $x^2 = 0$.

Exercise 1. Sketch the curves $xy = 0$ and $x^2 = 0$.

The goal of this worksheet is to completely classify conics up to equivalence (working over $\mathbb{C}$). Two conics are said to be equivalent if they are related by an invertible affine linear change of coordinates and a nonzero scaling; for example, the change of coordinates $X = x + y$, $Y = x - y$ turns $x^2 - y^2 = 9$ into $XY = 9$. Scaling by a factor of $1/3$ turns this into $\frac{1}{3}XY = 3$. We will make successive changes of coordinates/scalings to eliminate/simplify terms in P, putting it into one of a list of possible "normal forms".

Exercise 2. Suppose $a \neq 0$. Consider the new coordinates $X = x + \frac{by}{2a}$, $Y = y$. Rewrite $P(x, y)$ in terms of X and Y (hint: first write x and y in terms of X and Y and then substitute back). Hopefully you now have something with no XY term. How could we eliminate this term if $a = 0$? **From now on, we assume WLOG that $b = 0$.**

Exercise 3. If $a \neq 0$, find a change of coordinates to get rid of d. (Hint: Complete the square...) Similarly for $c \neq 0$ eliminate e.

As $b = 0$, we know that either $a \neq 0$ or $c \neq 0$. If $a = 0$, $c \neq 0$ we can switch x and y to get $a \neq 0$, $c = 0$. Scaling by $1/a$ we can assume $a = 1$. We have arrived at one of the following possibilities:

$$x^2 + cy^2 + f = 0 \quad c \neq 0$$
$$x^2 + ey + f = 0$$

Exercise 4. Suppose $P(x, y) = x^2 + cy^2 + f$, $b \neq 0$. In each case below, find a change of coordinates putting P in the desired form (up to scale):

- If $f \neq 0$, get to $X^2 + Y^2 = 1$.
- If $f = 0$, get to $X^2 + Y^2 = 0$.

Exercise 5. Suppose $P(x, y) = x^2 + ey + f = 0$. If $e = f = 0$ then the equation is $x^2 = 0$. In each of the remaining cases, find a change of coordinates putting P into the desired form (up to scale):

- If $e \neq 0$, get to $Y = X^2$.
- If $e = 0$ and $f \neq 0$, get to $X^2 = 1$.

This gives us a complete list of normal forms for conics (over $\mathbb{C}$):

A.1	$x^2 + y^2 = 1$	
A.2	$y = x^2$	
B.1	$x^2 + y^2 = 0$	two transverse lines ($x = \pm iy$)
B.2	$x^2 = 1$	two parallel lines ($x = \pm 1$)
C	$x^2 = 0$	double line($x = 0$)

Exercise 6 (Optional bonus question)**.** Over $\mathbb{R}$, we can only take square roots of positive numbers. Show that we get some extra cases:

A.1.a	$x^2 + y^2 = 1$	ellipse
A.1.b	$x^2 - y^2 = 1$	hyperbola
A.1.c	$x^2 + y^2 = -1$	empty
A.2	$y = x^2$	parabola
B.1.a	$x^2 + y^2 = 0$	single point($x = y = 0$)
B.1.b	$x^2 - y^2 = 0$	two transverse lines ($x = \pm y$)
B.2.a	$x^2 = 1$	two parallel lines ($x = \pm 1$)
B.2.b	$x^2 = -1$	empty
C	$x^2 = 0$	double line($x = 0$)