Algebraic Curves: Worksheet 3

Singularities

	D	C	B	B+/A-	A
γ	all	all	all	all	all
β	0	1	1	2	2
α	0	0	1	1	2

γ **Exercise 1.** Find any singularities of $\{xy^2 - x^4 = 0\}$. For each singularity, calculate its multiplicity and the tangencies.

γ **Exercise 2.** Prove that the curve $\{x^3 + y^3 = 1\}$ is smooth.

γ **Exercise 3.** This is a slightly woolly question because we haven't defined intersection multiplicity beyond the rough idea of "the number of points that appear after a small deformation". We will return to this question once we've covered the relevant theory. In each case below, give an opinion[1] about whether you think a pair of curves exist having the stated properties. If you think it exists, draw it.

(a) A line C and a conic C' intersecting at a single point p with $i_p(C, C') = 2$.

(b) A line C and a cubic C' intersecting at a single point p with $i_p(C, C') = 3$.

(c) Two lines C, C' intersecting at a single point p with $i_p(C, C') = 2$.

(d) Two conics intersecting transversely at n points for each of $n = 4, 3, 2, 1$.

(e) Two cubics intersecting at nine points.

(f) A quartic curve C and a line C' such that C and C' intersect at two points p, q with $i_p(C, C') = i_q(C, C') = 2$.

γ **Exercise 4.** An ideal I in a ring R is said to be *prime* if, for all $a, b \in R$, the product ab is in I if and only if $a \in I$ or $b \in I$. Show that if $f \in R$ is a prime element then the ideal (f) is prime.

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[1] This opinion should be based on your efforts, doomed or otherwise, to construct the curves, so make sure you hand in *all* working! Marks will be awarded for effort.

β **Exercise 5.** The *capricornoid curve* is the plane curve C defined by the equation:

$$x^2(x^2+y^2) = 2(y-x^2-y^2)^2.$$

(a) Find the singularities of C which lie on the line $x=0$.

(b) For each of these singularities, calculate the multiplicity and find the tangents of C.

(c) To convert this into an α-question, prove that *all* the singularities of C satisfy $x=0$. (This involves more stamina and mess than ingenuity, but stamina should be rewarded).

β **Exercise 6.** The *deltoid curve* is the plane curve D defined by the equation

$$(x^2+y^2)^2 + 2(x^2+y^2) - 1/3 = 8/3(x^3-3xy^2).$$

(a) Find the singularity of D which lies on the line $y=0$.

(b) Calculate the multiplicity of this singularity and find the tangents of D at this point.

(c) To convert this into an α-question, find the remaining singularities of D.

β **Exercise 7.** Given[2] a finite set of points

$$p_1 = (a_1, b_1), \ldots, p_N = (a_N, b_N) \in \mathbf{A}^2(k)$$

write down a polynomial $g(x,y)$ such that $g(p_1)=1$ and $g(p_2)=\cdots=g(p_N)=0$. (Hint: Try it for two points first. Then try it for three points. Then see if you can do it for any number of points.) How would you modify your argument to work for points in $\mathbf{A}^n(k)$?

β **Exercise 8.** This builds on Exercise 4. Prove that if I is a prime ideal then $\mathrm{rad}(I)=I$ (Hint: Use the Nullstellensatz.)

α **Exercise 9.** Consider the ideal $(x^2,y^2) \subset k[x,y]$. The quotient ring $k[x,y]/(x^2,y^2)$ is a vector space over k. Explain why (the equivalence classes) $1, x, y, xy$ form a basis for this vector space. (Why do they span? Why are they linearly dependent?)

α **Exercise 10.** Find a set of six monomials which span the quotient ring

$$k[x,y]/(y^2-x^3, x^2+y^2-1).$$

Prove that they span, but do not try to prove linear independence!

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[2]Sometimes people misinterpret questions like this. I don't mean "pick a particular set of points for yourself and then find a polynomial with these properties"; I mean "if, after you'd answered this question, *I* were to give you some randomly chosen set of points, your answer to this question would produce for me a polynomial with the desired properties with respect to those points."

α **Exercise 11.** Let a, b, c, d, e, f be numbers and consider the conic

$$C = \{ax^2 + bxy + cy^2 + dx + ey + f = 0\}.$$

Show that (x, y) is a singularity of C if and only if

$$\begin{pmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Deduce that if C is singular then $\det \begin{pmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{pmatrix} = 0$. Is the converse true?

α **Exercise 12.** This builds on Exercises 4 and 4. In this exercise, we will show that an algebraic set X is irreducible if and only if $\mathbb{I}(X)$ is a prime ideal.

(a) Suppose that an algebraic set X can be written as $\mathbb{V}(J) \cup \mathbb{V}(K)$ with $\mathbb{V}(J) \subsetneq X$ and $\mathbb{V}(K) \subsetneq X$. Show that $\mathbb{I}(X)$ is not prime. (Hint: You're trying to find polynomials $f_1 \in J$ and $f_2 \in K$ such that $f_1 \notin \mathbb{I}(X)$, $f_2 \notin \mathbb{I}(X)$ but $f_1 f_2 \in \mathbb{I}(X)$.)

(b) Suppose that X is an algebraic set and $\mathbb{I}(X)$ is not prime. Show that there are algebraic sets Y and Z with $X = Y \cup Z$ and $Y, Z \neq X$.
