Algebraic Curves: Worksheet 4

Intersections

	D	C	B	B+/A-	A
γ	all	all	all	all	all
β	0	1	1	2	2
α	0	0	1	1	2

γ **Exercise 1.** Let $I \subset k[x,y]$ be an ideal. For each polynomial $f \in k[x,y]$, recall that $\hat{f}\colon k[x,y]/I \to k[x,y]/I$ denotes the linear map

$$\hat{f}g = fg.$$

Given two polynomials $f_1, f_2 \in k[x,y]$, prove **three** of the following properties of $\hat{f}_1$ and $\hat{f}_2$:

(a) if $f_1 = f_2 \mod I$ then $\hat{f}_1 = \hat{f}_2$.

(b) $\widehat{f_1 f_2} = \hat{f}_1 \hat{f}_2$.

(c) $\widehat{f_1 + f_2} = \hat{f}_1 + \hat{f}_2$.

(d) $\hat{f}_1 \hat{f}_2 = \hat{f}_2 \hat{f}_1$

(e) $\hat{f}_1 = f_1(\hat{x}, \hat{y})$.

γ **Exercise 2.** Decompose the ring $k[x,y]/(y-1, y-x^2)$ into simultaneous generalised eigenspaces for the action of $\hat{x}$ and $\hat{y}$. Sketch the curves $\{y = 1\}$ and $\{y = x^2\}$ and explain the results of your eigenspace calculation in terms of intersections. You may assume that 1 and x form a monomial basis for the quotient ring. *You can upgrade this to a β question by explaining why 1 and x form a basis for the quotient ring.

γ **Exercise 3.** Let $M\colon V \to V$ be a linear map from a vector space V to itself. Suppose that V_λ is a nonzero generalised eigenspace with eigenvalue λ. Show that V_λ contains at least one honest eigenvector (i.e. a vector such that $(M - \lambda \mathrm{id})v = 0$). (Hint: Pick a nonzero generalised eigenvector v. Why must one of the vectors $(M - \lambda \mathrm{id})^n v$ be an eigenvector?)

β **Exercise 4.** Decompose the ring $k[x,y]/(3x-2y-1, y^2-x^3)$ into simultaneous generalised eigenspaces for the action of $\hat{x}$ and $\hat{y}$. Sketch the curves $\{3x = 2y+1\}$ and $\{y^2 = x^3\}$ and explain the results of your eigenspace calculation in terms of intersections. You may assume that $1, x, x^2$ form a monomial basis for the quotient ring.

β **Exercise 5.** Let $R = k[x_1, \ldots, x_n]$ and $I \subset R$ be an ideal with $S := R/I$ finite-dimensional. Let $S = \bigoplus_{\boldsymbol{a} \in \mathbb{V}(I)} S_{\boldsymbol{a}}$ be the decomposition of S into simultaneous generalised eigenspaces for $\hat{x}_1, \ldots, \hat{x}_n$. Show that if $\boldsymbol{a} \neq \boldsymbol{b}$ and $g_1 \in S_{\boldsymbol{a}}$ and $g_2 \in S_{\boldsymbol{b}}$ then $g_1 g_2 = 0$. (Hint: Show that $g_1 g_2 \in S_{\boldsymbol{a}} \cap S_{\boldsymbol{b}}$. This intersection is 0 because the sum is a direct sum.)

β **Exercise 6.** We can think of a polynomial $f = \sum_{i,j} a_{ij} x^i y^j \in k[x,y]$ as a polynomial $\sum_j \alpha_j y^j$ in y with coefficients $\alpha_j = \sum_i a_{ij} x^i$ in $k[x]$ (i.e. as an element of $k[x][y]$. We say that f is *primitive* if its coefficients $\alpha_0, \ldots, \alpha_n$ are coprime (as polynomials). Show that if $f = \sum_j \alpha_j y^j \in k[x][y]$ and $g = \sum \beta_j y^j \in k[x][y]$ are primitive then fg is primitive. (Hint: Suppose $h \in k[x]$ is a nonconstant polynomial dividing fg and let r, s be the smallest numbers such that h divides neither α_r nor β_s. Consider the coefficient of y^{r+s} in fg to get a contradiction.)

β **Exercise 7.** Suppose that $f(x,y,t)$ is a polynomial in three variables. This defines for us a *family of curves* thinking of t as a parameter:

$$C_t := \{(x,y) \,:\, f(x,y,t) = 0\} \subset \mathbf{A}^2.$$

The *envelope* of the family C_t is defined to be the set of points

$$\left\{(x,y) \in \mathbf{A}^2 \,:\, \exists t \text{ such that } f(x,y,t) = 0 \text{ and } \frac{\partial f}{\partial t}(x,y,t) = 0\right\}.$$

Suppose u, v, h are positive real numbers with $v > u$ and let

$$f(x,y,t) = (x-vt)^2 + y^2 - u^2t^2 + h^2\}.$$

(a) Fix $u = h = 1$ and $v = 2$ and sketch some of the curves C_t in the family.

(b) Find the envelope (with u, v, h arbitrary) by eliminating t from the equations $f = 0$ and $\partial f / \partial t = 0$.

(c) Again for $u = h = 1$ and $v = 2$, sketch the envelope (superimposed on your picture of the curves C_t).

(d) The numbers u, v, h are supposed to represent (respectively) the speed of sound, the speed of a supersonic jet, and the height of the jet above ground level. Why is the envelope you calculated called the *shockwave*? (Hint: Draw a picture of the jet flying and use Pythagoras's theorem.)

α **Exercise 8.** Let C be the line $\{y = 0\}$ and let $C' = \{f = 0\}$ be another curve such that y does not divide f. Let $\phi(x)$ be the polynomial $f(x, 0)$ and suppose that $\phi(x)$ factors as $(x-a_1)^{m_1} \cdots (x-a_k)^{m_k}$. Explain why the intersections $C \cap C'$ are in one-to-one correspondence with the roots $a_1, \ldots, a_k$ of ϕ and prove that $i_{(a_j,0)}(C, C') = m_j$ for all j. (Hint: Let $\phi_j = \prod_{l \neq j}(x - a_l)^{m_l}$ and consider the polynomials $\phi_j,\ (x - a_j)\phi_j, \ldots, (x - a_j)^{m_j-1}\phi_j$.)

α **Exercise 9.** Let $M_1, \ldots, M_n \colon V \to V$ be linear maps which commute with one another. Show that each simultaneous generalised eigenspace contains an honest simultaneous eigenvector, i.e. a vector v such that $M_1 v = \lambda_1 v, \ldots, M_n v = \lambda_n v$. (Hint: Do Exercise 3 first. This question is best done by induction on n. One key point of the inductive step is to show that the simultaneous honest eigenspaces for $M_1, \ldots, M_{n-1}$ are preserved by M_n.)

α **Exercise 10.** Let $f(x, y)$ be a polynomial and $C = \{f = 0\}$ be the corresponding curve in $\mathbf{A}^2$. Let $p = (0, 0)$ and suppose that $p \in C$. Suppose that $\tilde{x}$ and $\tilde{y}$ are new coordinates related to x, y by

$$x = a\tilde{x} + b\tilde{y}, \qquad y = c\tilde{x} + d\tilde{y}.$$

(Note that $ad - bc \neq 0$ so that this change of coordinates is invertible). Recall that the multiplicity $m_p(C)$ is defined to be the smallest d such that the dth order part of the Taylor series of f at p is nonzero. Implicit in this definition is a choice of coordinates: we will see that a different choice will yield the same answer.

(a) Let $\tilde{m}_p(C)$ be the multiplicity as computed using the $\tilde{x}, \tilde{y}$ coordinates. Show that

$$m_p(C) \leq \tilde{m}_p(C).$$

(b) Since $ad - bc \neq 0$, why does this imply $m_p(C) = \tilde{m}_p(C)$?

α **Exercise 11.** Let $C = \{f = 0\}$ and $C' = \{g = 0\}$ be curves and let $p \in C \cap C'$ be a point with

$$m_p(C) = c, \qquad m_p(C') = c'.$$

Suppose that $L = \{\ell = 0\}$ is a line which is tangent to both C and C' at $p \in C \cap C'$. Show that there exist nonzero polynomials $\tilde{f}$ and $\tilde{g}$ of degrees $c-1$ and $c'-1$ respectively such that $\tilde{g}f - \tilde{f}g = 0 \mod \mathfrak{m}^{c+c'}$, where $\mathfrak{m} = (x, y)$. Deduce that $i_p(C, C') > cc'$. (Hint: Look at the proofs of Propositions 10.10 and 10.11).

α **Exercise 12.** Following on from Exercise 7, compute the envelope of the family of lines

$$C_t = \{(x, y) \,:\, y = tx + t^3\}.$$

Sketch both the envelope and some of the lines in the family. Find the singularity of the envelope.

Envelopes of families of straight lines are important in the study of optics. For example, envelopes of reflected light rays are the bright curves you see at the bottom of cups, also known as "caustics". These caustics often have singularities. Caustics are also responsible for rainbows.
