Algebraic Curves: Worksheet 5

	D	C	B	B+/A-	A
γ	all	all	all	all	all
β	0	1	1	2	2
α	0	0	1	1	2

γ **Exercise 1.** For each polynomial f below:

(a) write down the homogenisation F,

(b) find the intersection points of the projective curve $C := \{F = 0\}$ with the line L at infinity,

(c) by working in a different affine chart (like $x = 1$ or $y = 1$) compute the multiplicities $i_p(L, C)$ of any intersections you found in (b),

(d) sketch the (real) affine curve $\mathbb{V}_{\mathbb{R}}(f)$ and its asymptotes.

Here are the curves:

(i) $f(x, y) = y^2 - x^3$,

(ii) $f(x, y) = x^2y - xy^2 - 1$,

(iii) $f(x, y) = x^3 - 3x^2 + 2x$.

γ **Exercise 2.** Look back at Sheet 3, Question 1.3. If there were any configurations of curves which you were unable to draw back then, either draw them now or prove that they cannot exist. (Hint: To construct affine curves C and C' intersecting in *fewer* points than allowed by Bézout's theorem, you might like to take curves which intersect in the *right* number of points and then change coordinates to ensure the line at infinity passes through some of the intersections.) Do the same for the following configurations:

(i) Two affine cubics C, C' intersecting at four points each with multiplicity 2.

(ii) An irreducible conic C passing through five points all of which lie on a line C'.

(iii) An irreducible cubic C passing through six points all of which lie on a conic C'.

(iv) An irreducible cubic C passing through seven points all of which lie on a conic C'.

(v) A reducible cubic C passing through seven points all of which lie on a conic C'.

β **Exercise 3.** Following the method suggested in the notes on "existence of curves with constraints", find a cubic with singularities at $(0,0)$, $(1,0)$ and $(0,1)$. (Note: The purpose of this question is to show you understand the method in the notes, so you won't get any credit for finding a curve in any other way, e.g. guessing).

β **Exercise 4.** Recall Harnack's theorem: *Let C be an irreducible plane curve of degree d and let $M = 1 + \frac{1}{2}(d-1)(d-2)$. If C has M ovals then it cannot have any other connected components (oval or not).*

In the notes, we proved Harnack's theorem for cubics and quartics. Mimicking the argument for quartic curves, give a proof that works for curves of any degree. *Hint:* $M + (d-3) = \frac{d'(d'+3)}{2}$ *for* $d' = d - 2$.

β **Exercise 5.** Show that a polynomial is homogeneous of degree d if and only if it is a sum of monomials of degree d.

β **Exercise 6.** Suppose that $F(X_1, \dots, X_n)$ is a homogeneous polynomial of degree d. Show that

$$X_1 \frac{\partial F}{\partial X_1} + \cdots + X_n \frac{\partial F}{\partial X_n} = d\,F.$$

α **Exercise 7.** (Builds on Exercise 6.) Let $F(X,Y,Z)$ be a homogeneous polynomial of degree $d \geq 1$. Show that a point $P \in \mathbb{P}^2(k)$ is a singular point of the curve $C = \{F = 0\}$ if and only if

$$\frac{\partial F}{\partial X}(P) = \frac{\partial F}{\partial Y}(P) = \frac{\partial F}{\partial Z}(P) = 0.$$

(In particular, we don't need to check that $P \in C$).

α **Exercise 8.** (a) Let $f(x,y)$ be a polynomial and $p = (a,b)$. How many constraints are imposed by the vanishing at p of f and all its derivatives of order strictly smaller than k?

(b) In the notes we showed that through any configuration of $d(d+3)/2$ points we can find a curve of degree d. You can think of Exercise 3 as an example of a more general result, which finds curves through points with at least a specified multiplicity. Formulate and prove as general result as you can about the existence of curves passing through points with specified multiplicities.

α **Exercise 9.** Given $d+1$ numbers $\lambda_0, \lambda_1, \dots, \lambda_n$, define the *Vandermonde matrix*

$V(\lambda_0, \dots, \lambda_d)$ to be the $(d+1)$-by-$(d+1)$ matrix

$$\begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^d \\ 1 & \lambda_1 & \cdots & \lambda_1^d \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^d \end{pmatrix}.$$

We will show that

$$\det(V(\lambda_0, \dots, \lambda_d)) = \prod_{d \geq i > j \geq 0} (\lambda_i - \lambda_j). \tag{1}$$

(a) Subtract the first row from the other rows.

(b) For each row, all the entries have a common factor (possibly a different one for each row). Find these common factors.

(c) Using (a) and (b) and the basic properties of determinants from MATH105, show that

$$\det(V(\lambda_0, \dots, \lambda_d)) = \left(\prod_{i \neq 0} (\lambda_i - \lambda_0) \right) \det(V(\lambda_1, \dots, \lambda_d)).$$

(d) Prove Equation (1) by induction.

α **Exercise 10.** The goal of this question is to prove that if $\gcd(f, g) = 1$ then the curves $\{f = 0\}$ and $\{g = 0\}$ have only a finite number of intersection points. We quietly assumed this throughout our development of intersection theory, so until we have proved it, our proof of Bézout's theorem is incomplete. We will need some ideas from MATH322, which we briefly recap here (you should be able to do the question even without taking MATH322).

Let $R = k[x]$ be the ring of polynomials in x and $K = k(x)$ the field of rational functions; since polynomials are just rational functions with denominator 1, we can think of R as a subset of K. We can think of a polynomial $f \in k[x, y]$ as an element of $R[y]$ by grouping the terms; for example,

$$3x^2y^2 + 2xy^2 - 4x^2 + y + 2 = (3x^2 + 2x)y^2 + y + (2 - 4x^2).$$

If $f \in R[y]$, let $c(f) \in R$ denote the polynomial which is the greatest common divisor of the coefficients, and write $f = c(f) \cdot f_{prim}$. We say that f_{prim} is the *primitive part* of f; is it uniquely determined up to units. We say that f is primitive if $c(f)$ is constant. *Gauss's Lemma* is the statement that (up to units) the primitive part of a product of polynomials equals the product of the primitive parts (this is proved in MATH322; you may use it here).

(a) Going further, we can think of a polynomial $f \in k[x, y]$ as an element of $K[y]$. Explain why we can write any $h \in K[y]$ in the form $h = \frac{a}{b} h_{prim}$ where $a, b \in R$ and h_{prim} is primitive and $\gcd_R(a, b) = 1$.

(b) Suppose that $f, g \in R[y]$ have $\gcd_{R[y]}(f, g) = 1$, in other words that the only polynomials $h \in R[y]$ which divide both are constant. Show that $\gcd_{K[y]}(f, g) = 1$, in other words, the only polynomials $h \in K[y]$ which divide both f and g are constant in y (i.e. just elements of K).

(c) The advantage of working in $K[y]$ is that $K[y]$ is a Euclidean domain, that is Euclid's algorithm for finding the gcd works. In a Euclidean domain, we have *Bézout's lemma*[1], which asserts that there exist elements $a, b \in K[y]$ such that $af + bg = \gcd_{K[y]}(f, g)$. If $\gcd_{K[y]}(a, b) = 1$, why does this imply the existence of polynomials $\alpha, \beta \in k[x, y]$ and $\delta \neq 0 \in k[x]$ such that $\alpha f + \beta g = \delta$?

(d) Deduce that if $\gcd_{k[x,y]}(f, g) = 1$ then there are only a finite number of possible values for the x-coordinate of an intersection point in

$$\{f = 0\} \cap \{g = 0\}.$$

How could we now show that there are only finitely many intersection points?

α **Exercise 11.** Let C be a curve, and suppose $p \in C$ is a smooth point. Let T_pC denote the tangent line to C at p. Since C and T_pC do not intersect transversely at p, we have $i_p(C, T_pC) \geq 2$. A *flex* of C is a point $p \in C$ with $i_p(C, T_pC) \geq 3$.

(a) Show that the origin is a flex of $\{y = x^3\}$.

(b) Can an irreducible conic have any flexes?

(c) Let $C = \{f = 0\}$ and suppose $p = (0, 0)$ is a smooth point of C. Show that p is a flex of C if

$$2f_{xy}f_xf_y = f_{xx}f_y^2 + f_{yy}f_x^2$$

where subscripts denote derivatives, e.g. $f_{xy} = \partial^2 f/\partial x\partial y$, and all derivatives are evaluated at p. (Hint: Sheet 4, Exercise 8 showed[2] that if (at, bt) is a parametrisation of a line L then $i_{(0,0)}(\{f = 0\}, L)$ is the multiplicity of 0 as a root of $f(at, bt)$. What is a parametrisation of the tangent line T_pC?)

α **Exercise 12.** (Builds on Exercises 6 and 11.) Suppose that $C = \{F = 0\}$ is a projective curve defined by a homogeneous polynomial $F(X, Y, Z)$. Show that a smooth point $p \in C$ is a flex of C if and only if

$$\Delta := \det \begin{pmatrix} F_{XX} & F_{YX} & F_{ZX} \\ F_{XY} & F_{YY} & F_{ZY} \\ F_{XZ} & F_{YZ} & F_{ZZ} \end{pmatrix} (p) = 0.$$

(Hint: Use Exercise 6 and 11. Be prepared for some mess with matrices (e.g. row operations that leave the determinant unchanged).)

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[1] This is a different, much easier, result than Bézout's theorem! It is proved in MATH322.

[2] Technically the proof there was for the line $L = \{y = 0\}$, but works in general after a change of coordinates.