Lecture 17: Gauge fixing I

Jonathan Evans

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Our aim today is to prove

Theorem (Uhlenbeck)

Consider the trivial $U(n)$-bundle over the unit 2-disc. There exist $\kappa > 0$ and $c < \infty$ such that if $\nabla = d + A$ is a connection with $||F_\nabla||_{L^2} \leq \kappa$ then there is an L^2_2-gauge transformation u such that $u\nabla = d + A'$ satisfies

- $d^*A' = 0$,
- $||A'||_{L^2_1} \leq c||F_{\nabla'}||_{L^2}$.

Moreover if we require the boundary condition that the radial component $\sum_i (x_i/r)A_i$ tends to zero as $r \to 1$ then the resulting A' is unique up to constant gauge transformations.

As when we proved that a unitary connection induces the structure of a holomorphic vector bundle, it will be convenient to work over a compact space so we use S^2 instead of the 2-disc (in this we follow Donaldson-Kronheimer rather than Uhlenbeck's original proof).

Theorem

There are constants ζ and N such that if $\nabla_t = d + A_t$ is a path of connections on the trivial $U(n)$-bundle over S^2 with $A_0 = 0$ and $||F_{\nabla_t}||_{L^2} \leq \zeta$ then there exist L^2_2-gauge transformations u_t such that $u_t\nabla_t = d + A'_t$ satisfy $d^*A'_t = 0$ and $||A'_t||_{L^2_1} < 2N||F_{\nabla_t}||_{L^2}$ (unless $A'_t = 0$).

Let's write $S \subset [0,1]$ for the subset of the interval consisting of t for which the conclusion of the theorem holds.

To deduce Uhlenbeck's theorem from this we take our connection of interest ∇ over the disc and construct a connection on the sphere by pulling back along the map p which collapses the 2-sphere onto a disc (by projecting onto a plane through the equator). Obviously this is not smooth along the equator, so instead we take a sequence of smooth approximations p_ϵ to p which differ only in an ϵ neighbourhood of the equator. Now suppose ∇ has curvature L^2-bounded above by $\zeta/\sqrt{2}-\epsilon'$. Then for small enough ϵ, $p_\epsilon^*\nabla$ has curvature L^2-bounded above by ζ. Moreover we can define a path of connections ∇_t, $t \in [0,1]$ with $\nabla_0 = d$ by setting $A_t(x) = tA(tx)$ (here $x \mapsto tx$ is just rescaling the ball).

Since

$$\begin{aligned} ||F_{\nabla_t}||_{L^2} &= \sqrt{\int_{x\leq 1} |t^2 F_\nabla(tx)|^2 d\mathrm{vol}_x} \\ &= \sqrt{\int_{y\leq t} t^4 t^{-2} |F_\nabla|^2 d\mathrm{vol}_y} \\ &\leq t|F_\nabla|_{L^2} \\ &\leq \zeta/\sqrt{2} - \epsilon' \end{aligned}$$

we see that $p_\epsilon^* \nabla_t$ is a path of connections satisfying the hypotheses of the theorem. This allows us to put ∇ in Coulomb gauge on a slightly smaller ball, but that's all we need.

The idea will be to prove that S is both closed and open. Since $0 \in S$ the theorem will follow. First we prove a lemma.

Lemma

Let $d + A$ be a connection on the trivial bundle over S^2 such that $d^*A = 0$. Then there are constants $N, \eta > 0$ such that

$$||A||_{L^4} < \eta \implies ||A||_{L^2_1} \leq N||F_\nabla||_{L^2}.$$

Proof.

Since d is elliptic on $\ker(d^*)$ (think about it! This is just ellipticity of the Laplacian) and since it has no kernel (otherwise there would be nontrivial harmonic 1-forms on S^2 and hence a nontrivial class in $H^1(S^2)$) we have

$$||A||_{L^2_1} \leq C||dA||_{L^2}$$

Since $F_\nabla = dA + [A, A]$ and $||[A, A]||_{L^2} \leq ||A||^2_{L^4} \leq C'||A||_{L^4}||A||_{L^2_1}$ by Hölder and Sobolev we get

$$||A||_{L^2_1} \leq C||dA||_{L^2} \leq C||F||_{L^2} + CC'||A||_{L^4}||A||_{L^2_1}$$

and when $||A||_{L^4} < 1/(2CC') = \eta$ we can take the last term over to the other side and get

$$||A||_{L^2_1} \leq N||F||_{L^2}$$

(where $N = 2C$). □

Now we take $\zeta < \frac{\eta}{2CN}$ where η and N are given by this lemma and C is the Sobolev constant for the embedding $L^2_1 \subset L^4$.

Lemma

Given a path satisfying the hypotheses of the theorem, S is closed.

Proof.

Let $t_i \in S$ be a sequence converging to some t_∞ and write $A_{t_i} = A_i$, so that there exist gauge transformations u_i such that $A'_i = u_i A_i$ satisfies the conclusions of the theorem. Certainly as $i \to \infty$, $A_i \to A_{t_\infty} = A_\infty$. Since A'_i is bounded in L^2_1 there is a weakly convergent subsequence $A'_i \to A'_\infty$. We want to construct an L^2_2-gauge transformation u_∞ from A_∞ to A'_∞. But

$$A'_i = u_i^{-1} A_i u_i + u_i^{-1} du_i$$
$$\text{i.e. } du_i = u_i A'_i - A_i u_i$$

□

Proof, continued:

Since $u_i(x) \in U(n)$ which is compact we have $|u_i|_{L^2} \leq c$ and

$$|du_i|_{L^4} \leq c(|A_i'|_{L^4} + |A_i|_{L^4}) \leq cC(|A_i'|_{L^2_1} + |A_i|_{L^2_1})$$

Therefore $|u_i|_{L^4_1}$ is bounded uniformly in i and hence weakly converges to some u in L^4_1. We need to show that u is in L^2_2, but we know that

$$du = uA' - Au$$

and now I'm going to do something I said I wouldn't, which is to use another Sobolev theorem I haven't previously stated. It follows from Palais "Foundations of Global Analysis", Theorem 9.5(2) that $L^4_1 \otimes L^2_1 \to L^2_1$ is a well-defined Sobolev multiplication in 2-d. Therefore since $u \in L^4_1$ and $A, A' \in L^2_1$ we get $du \in L^2_1$ and hence $u \in L^2_2$. □

Proof.

The coclosedness equation is certainly preserved in the limit. It remains to show that the inequality (which is open!) is preserved. But if we know that $s \in [0,1]$ satisfies the hypotheses of the theorem then $||A'_s||_{L^4} \leq C||A'_s||_{L^2_1} < 2NC||F'_s||_{L^2} \leq \eta$ by the choice of ζ and hence by the lemma we first proved $||A'_s||_{L^2_1} \leq N||F'_s||_{L^2}$. This condition is closed and is strictly stronger than $||A_s||_{L^2_1} < 2N||F'_s||_{L^2}$ hence the open condition is preserved in the limit. □