Lecture 22: The Harder-Narasimhan stratification

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Last time we began to investigate the stratification on the space of connections given by $\mathcal{A}_\mu$, consisting of connections ∇ such that the associated holomorphic vector bundle $\mathcal{E}_\nabla$ has Harder-Narasimhan type μ. We partially ordered the μ and stated a theorem of Atiyah and Bott (originally Shatz) which told us that

$$\bar{\mathcal{A}}_\mu \subset \bigcup_{\lambda \geq \mu} \mathcal{A}_\lambda$$

This was the first of various conditions we need to use the stratification to compute the homology of $\mathcal{A}$. We'll sketch a proof of this fact first.

Atiyah-Bott-Shatz

The idea is clever. Remember in Donaldson's proof of the Narasimhan-Seshadri theorem we replaced the Yang-Mills functional

$$\int \mathrm{Tr}(F_\nabla \wedge \star F_\nabla)$$

by using a different norm (instead of Tr). Let $\phi\colon \mathfrak{g} \to \mathbb{R}$ be an adjoint-invariant convex function on the Lie algebra (the space of Hermitian matrices). Convex just means $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$ and define

$$\Phi(\nabla) = \int \phi(F_\nabla \wedge \star F_\nabla)$$

Since the adjoint-orbits of $U(n)$ on its Lie algebra are just the conjugacy classes of Hermitian matrices (and are therefore classified by their eigenvalues) an adjoint-invariant convex function can be thought of as a convex function on $\mathbb{R}^n$ (the space of eigenvalues).

Theorem

If ∇ is compatible with a holomorphic vector bundle with Harder-Narasimhan type $\underline{\mu}$ then $\Phi(\nabla) \geq \phi(\mu)$ (thinking of μ as the eigenvalues of a matrix).

The proof of this is very similar in spirit to the lower bounds on the Yang-Mills-like functional we proved during the proof of the Narasimhan-Seshadri theorem (specifically in Lecture 13). Now, for a holomorphic vector bundle E, we define $\Phi(E) = \inf_{\nabla \in \mathcal{O}(\mathcal{E})} \Phi(\nabla)$. Similarly to the N-S theorem it is possible to prove that for stable bundles

$$\Phi(E) = \phi(\mu)$$

(where μ is now the constant vector $(\mu(E), \ldots, \mu(E))$). That is there exists a constant central curvature connection. One can prove by induction using the HN filtration (and the filtration of semistable bundles with stable quotients) that

$$\boxed{\Phi(E) = \phi(\mu)}$$

for all bundles (where μ is now the HN type of E).

Now suppose that a connection ∇ in $\mathcal{A}_\lambda$ is in the closure of $\mathcal{A}_\mu$ (so $\mathcal{A}_\mu \ni \nabla_i \to \nabla$). Then because $\Phi(E_{\nabla_i})$ is defined by an infimum, $\phi(\lambda) = \Phi(E_\nabla) \geq \Phi(E_{\nabla_i}) = \phi(\mu)$. This is true for any convex invariant ϕ.

Theorem

Suppose that for any convex function on $\mathbb{R}^n$

$$\phi(\lambda) \geq \phi(\mu)$$

Then $\lambda \geq \mu$.

The proof of this requires no heavy machinery and I refer you to Section 12 of Atiyah and Bott. This shows that the closure of $\mathcal{A}_\mu$ is contained in the union of strata $\mathcal{A}_\lambda$ with $\lambda \geq \mu$.

Codimension

We also need to understand the codimension of $\mathcal{A}_\mu$ (to show that there are at most finitely many μ with a given codimension). At some point during the proof of Narasimhan-Seshadri we proved the following theorem of Atiyah-Bott.

Lemma

Fix an L^2_1-connection $\nabla' = \nabla + B$ (∇ is a smooth reference connection). The action $F : \mathcal{G}_{\mathbb{C}} \to \mathcal{A}$ (sending g to $g\nabla'$) of the L^2_2-complexified gauge transformations on the L^2_1-connections has the property that d_1F is Fredholm. Here $d_1F : L^2_2(\Omega^0(M; \mathrm{ad}(P_{\mathbb{C}}))) \to L^2_1(\Omega^1(M; \mathrm{ad}(P)))$ denotes the derivative at $1 \in \mathcal{G}_{\mathbb{C}}$.

Moreover we saw that $d_1F(\epsilon) = -(\nabla')^{0,1}\epsilon$ which is a compact perturbation of $\nabla^{0,1}$. In particular, the cokernel of d_1F (which constitutes a complement to the complexified gauge orbit of ∇) has codimension $H^1_{\bar\partial}(\mathrm{End}(\mathcal{E}))$ where this denotes the Dolbeault cohomology group. It is a happy fact of algebraic geometry that the Euler characteristic of Dolbeault cohomology for a holomorphic vector bundle V can be computed using the Riemann-Roch theorem

$$H^0_{\bar\partial}(V) - H^1_{\bar\partial}(V)) = 2c_1(V) + \mathrm{rank}(V)(2-2g)$$

where g is the genus of the curve. We will need this later. Now $\mathcal{A}_\mu$ is a union of complexified gauge orbits (those whose Harder-Narasimhan filtration has type μ). Let $\mathrm{End}_0\mathcal{E}$ denote the bundle of endomorphisms of $\mathcal{E}$ which preserve the filtration by subbundles $0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$ and consider the quotient bundle

$$0 \to \mathrm{End}_0(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \to \mathrm{End}_1(\mathcal{E}) \to 0$$

The corresponding Dolbeault LES reads

$$\cdots \to H^0(\mathrm{End}_1(\mathcal{E})) \to H^1(\mathrm{End}_0(\mathcal{E})) \to H^1(\mathrm{End}(\mathcal{E})) \to H^1(\mathrm{End}_1(\mathcal{E})) \to 0$$

since it's a bundle over a curve (complex dimension 1). We will show that $H^0(\mathrm{End}_1\mathcal{E}) = 0$. This group consists of Dolbeault cocycles, namely holomorphic endomorphisms! But there can be no nonzero such holomorphic endomorphisms. To see this, suppose $f\colon \mathcal{E} \to \mathcal{E}$ were such an endomorphism (nonzero). Since it does not preserve the HN filtration there exists a (minimal) k such that $f(\mathcal{E}_k) \not\subset \mathcal{E}_k$ and a minimal $\ell \geq k+1$ such that $f(\mathcal{E}_k) \subset \mathcal{E}_\ell$. By minimality of k and ℓ this descends to a nontrivial map $\mathcal{E}_k/\mathcal{E}_{k-1} \to \mathcal{E}_\ell/\mathcal{E}_{\ell-1}$. But this is a nontrivial map between semistable vector bundles where the target has strictly smaller slope. Therefore the kernel has strictly larger slope, contradicting semistability of $\mathcal{E}_k/\mathcal{E}_{k-1}$.

Therefore we have a SES

$$0 \to H^1(\mathrm{End}_0(\mathcal{E})) \to H^1(\mathrm{End}(\mathcal{E})) \to H^1(\mathrm{End}_1(\mathcal{E})) \to 0$$

we can think of $H^1(\mathrm{End}(\mathcal{E}))$ consisting of a) deformations of $\mathcal{E}$ which preserve the HN filtration ($H^1(\mathrm{End}_0(\mathcal{E}))$, tangent to $\mathcal{A}_\mu$) and b) deformations complementary to $\mathcal{A}_\mu$ (coming from $H^1(\mathrm{End}_1(\mathcal{E}))$). Indeed, this latter collection of deformation spaces have constant dimension, by Riemann-Roch and $H^0(\mathrm{End}_1(\mathcal{E})) = 0$.

This dimension is easily computed by Riemann-Roch because $c_1(A^* \otimes B) = c_1(B)\mathrm{rank}(A) - c_1(A)\mathrm{rank}(B)$. Therefore $\mathrm{End}(\mathcal{E})$ has $c_1 = 0$ and $c_1(\mathrm{End}_1(\mathcal{E})) = -c_1(\mathrm{End}_0(\mathcal{E}))$. This latter Chern class can be computed by choosing a Hermitian metric so that the HN filtration splits and $\mathrm{End}_0(\mathcal{E})$ is isomorphic (as a complex vector bundle) to the endomorphisms of $D_1 \oplus D_2 \oplus \cdots \oplus D_r$ which are upper triangular with respect to this decomposition, i.e.

$$\mathrm{End}_0(\mathcal{E}) \cong \bigoplus_{j>i} D_j^* \otimes D_i$$

which has $c_1 = \sum_{j>i} (\mathrm{rank}(D_i)c_1(D_j) - \mathrm{rank}(D_j)c_1(D_i))$. Overall we get

$$\dim(H^1(\mathrm{End}_1(\mathcal{E}))) = \sum_{i>j} ((n_i k_j - n_j k_i) + n_i n_j (g-1))$$

where $n_i = \mathrm{rank}(D_i)$ and $k_i = c_1(D_i)$. Since we can express the codimension entirely in terms of the vector $\underline{\mu}$ of slopes this means that our stratification satisfies the finiteness requirements outlined last lecture.