Lecture 24: Equivariant perfection and Morse strata

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In today's lecture we will see:

- why the stratification $\mathcal{A} = \bigcup_\mu \mathcal{A}_\mu$ is $\mathcal{G}_{\mathbb{C}}$-equivariantly perfect,
- why the Harder-Narasimhan stratification *should be* the Morse stratification (by unstable manifolds) of the space of connections with respect to the Yang-Mills function.

The contribution to the $\mathcal{G}_{\mathbb{C}}$-equivariant cohomology of $\mathcal{A}$ from $\mathcal{A}_\mu$ is determined by the normal bundle N_μ of $\mathcal{A}_\mu \subset \mathcal{A}$. Ideally we want to show that the cup product with the Euler class of the unit normal bundle (which is a sphere bundle) gives an injective map on the equivariant cohomology of $\mathcal{A}_\mu$. Notice that this is not quite the setting we were in before (we actually have a stratum which is a *union* of $\mathcal{G}_{\mathbb{C}}$-orbits) but this is still the right criterion.

Perfection

Recall from a couple of lectures ago: we showed that the Borel space $(\mathcal{A}_\mu)_{\mathcal{G}_\mathbb{C}}$ is homotopy equivalent to Borel space $(\mathcal{B}^0_\mu)_{\mathrm{Aut}(E^0_\mu)}$. Here E^0_μ really just means we've chosen a splitting $E = D_1 \oplus \cdots \oplus D_r$ (where D_i has slope μ_i) and we're looking at connections $\nabla \in \mathcal{B}^0_\mu$ compatible with a holomorphic vector bundle structure such that the D_i are semistable holomorphic subbundles and complexified gauge transformations $g \in \mathrm{Aut}(E^0_\mu)$ such that g preserves each D_i. We may similarly reduce from N_μ (whose fibre is $H^1(M; \mathrm{End}_1(E))$) to $N^0_\mu = N_\mu|_{\mathcal{B}^0_\mu}$.

Just to make things slightly easier, let's recall that $\mathcal{G}_0$, the subgroup of gauge transformations which equal the identity at some basepoint, acts freely on $\mathcal{A}$ and hence

$$\mathcal{A} \times_{\mathcal{G}} E\mathcal{G} \simeq (\mathcal{A}/\mathcal{G}_0) \times_G EG$$

i.e. $H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}) = H^*_G(\mathcal{A}/\mathcal{G}_0)$, where $G = U(n)$. In our setting we get

$$H^*_{\mathrm{Aut}(E^0_\mu)}(\mathcal{B}^0_\mu) = H^*_{U(\mu)}(\mathcal{B}^0_\mu/(\mathcal{G}_0 \cap \mathrm{Aut}(E^0_\mu)))$$

where $U(\mu)$ is just subgroup of unitary matrices preserving the splitting $D_1 \oplus \cdots \oplus D_r$ at the basepoint. Now inside $U(\mu)$ there is a torus $T(\mu)$ of block-diagonal gauge transformations which are constant and each block is just a scalar. Since each block is a constant gauge transformation on some D_i these transformations fix connections in $\mathcal{B}^0_\mu$. In particular, the normal bundle to $B^0_\mu/(\mathcal{G}_0 \cap \mathrm{Aut}(E^0_\mu)) \subset \mathcal{A}_\mu/\mathcal{G}_0$ is acted upon by $T(\mu)$. To understand the equivariant Euler class of the normal bundle, it suffices to understand the $T(\mu)$-equivariant Euler class for the following simple reason:

Lemma

$H^*_{U(\mu)}(X) \hookrightarrow H^*_{T(\mu)}(X)$ for any $U(\mu)$-space X.

Let's prove it for $U(n)$ and T^n, the general case is an application of Künneth. We'll define an inverse for $f^*\colon H^*(X_{U(n)}) \to H^*(X_{T^n})$ using fibre integration on

$$U(n)/T^n \to X_{T^n} \xrightarrow{f} X_{U(n)}$$

Let $\xi \subset TX_{T^n}$ be the vertical tangent bundle (of vectors tangent to fibres) and define $\tau\colon H^*(X_{T^n}) \to H^*(X_{U(n)})$ by

$$\tau(x) = f_!(x\chi(\xi))$$

where χ denotes the Euler class. Since $f_!(f^*x \cup y) = xf_!(y)$ (should be obvious when you think of $f_!$ as a fibre integral) we have

$$\tau \circ f^*(x)) = x \int_{U(n)/T^n} \chi$$

but the multiplier is just the Euler number of $U(n)/T^n$ which is $n!$. Therefore $\tau/n!$ is an inverse for f^*.

Exercise

By considering $U(n-1)\times U(1)\subset U(n)$ prove by induction that $\chi(U(n)/T^n)=n!$.

So we have to understand the action of $T(\mu)$ on $H^1(M;\mathrm{End}_1(E))$. Remember that $\mathrm{End}_1(E)$ is the quotient of $\mathrm{End}(E)=\sum_{i,j}\mathrm{Hom}(D_i,D_j)$ by the subbundle of endomorphisms preserving the filtration, i.e. $\mathrm{End}_0(E)=\sum_{i\geq j}\mathrm{Hom}(D_i,D_j)$. Therefore $\mathrm{End}_1(E)=\sum_{i<j}\mathrm{Hom}(D_i,D_j)$. On $\mathrm{Hom}(D_i,D_j)=D_i^*\otimes D_j$ the element $(t_1,\ldots,t_r)\in T(\mu)$ acts by $t_i^{-1}t_j$ and hence by the same character on the cohomology group we're interested in. Now since every connection in $\mathcal{B}^0_\mu$ is fixed by $T(\mu)$ we have

$$H^*_{T(\mu)}(\mathcal{B}^0_\mu/(\mathcal{G}_0\cap\mathrm{Aut}(E^0_\mu)))\cong H^*(\mathcal{B}^0_\mu/(\mathcal{G}_0\cap\mathrm{Aut}(E^0_\mu)))\otimes H^*(BT(\mu))$$

By Künneth's theorem we can write the Euler class in the form

$$1 \otimes a_0 + \sum b_i \otimes a_i$$

where $b_i \in H^*(\mathcal{B}^0_\mu / (\mathcal{G}_0 \cap \mathrm{Aut}(E^0_\mu)))$ has degree bigger than 0.

Exercise

If $a_0 \neq 0$ show it is impossible for the Euler class to be a zero divisor.

The interpretation of a_0 is just the equivariant Euler class of the vector bundle $N^0_\mu|_\star$ where $\star \in \mathcal{B}^0_\mu$ is a point. So we restrict attention to the representation of $T(\mu)$ on the vector space $H^1(M; \mathrm{End}_1(E))$.

Since $BT(\mu) = (\mathbb{CP}^\infty)^r$ with generators a_i (corresponding to the standard representation of the ith component) and the normal bundle is just a direct sum $\sum_{i<j}(N^0_\mu)_{i,j}$ where $(N^0_\mu)_{i,j}$ is a representation with weight $t_i^{-1}t_j$, the top Chern class (Euler class) of this bundle is the product over $i<j$ of the top Chern classes of $(N^0_\mu)_{i,j}$ which gives

$$\prod_{i<j}(a_j - a_i)^{p_{i,j}}$$

where the $p_{i,j}$ depends on the rank of $(N^0_\mu)_{i,j}$. This is just because the vector space splits into 1-dimensional irreps which are tensor products of reps corresponding to the characters t_i^{-1} and t_j and the first Chern class of a tensor product is a sum. In particular it's not zero and since $H^*(BT(\mu))$ is a polynomial ring in the a_i, multiplication by the Euler class is injective on equivariant cohomology.

Morse strata

Let me explain why one should believe that the Harder-Narasimhan strata are the unstable manifolds for the upward gradient flow of the Yang-Mills functional. First, we'll show that the gradient flow of the Yang-Mills functional is tangent to $\mathcal{G}_{\mathbb{C}}$-orbits. Recall that

$$\mathcal{YM}(\nabla + ta) = \mathcal{YM}(\nabla) + 2t \int \star F_{\nabla} \wedge \star \nabla a + \mathcal{O}(t^2)$$

and hence the gradient is

$$\mathrm{grad}(\mathcal{YM}) = \star\nabla \star F_{\nabla}$$

(critical points are Yang-Mills connections!). This makes sense because a tangent vector to $\mathcal{A}$ should be a 1-form with values in $\mathrm{ad}(P)$ and that's what this is.

The tangent space to a gauge orbit ($\mathcal{G}$-orbit) at ∇ is the space of $\nabla\alpha$, $\alpha \in \Omega^0(M; \mathrm{ad}(P))$. This is because $\exp(\epsilon\alpha)\nabla = \nabla - (\nabla \exp^{\epsilon\alpha}) \exp^{-\epsilon\alpha} = \nabla - \epsilon\nabla\alpha + \mathcal{O}(\epsilon^2)$. Remember that to define the complexified action, we identify the space of 1-forms with values in $\mathrm{ad}(P)$ with the space of $(0,1)$-forms with values in $\mathrm{ad}(P_{\mathbb{C}})$ by sending

$$A \in \Omega^1(\mathrm{ad}(P)) \text{ to } A + i \star A$$

(remember that on a Riemann surface the complex structure acts on 1-forms by $\star$). Then an infinitesimal complexified gauge transformation

$$\beta = \alpha_1 + i\alpha_2 \in \Omega^0(M; \mathrm{ad}(P_{\mathbb{C}}))$$

gives the tangent vector

$$\begin{aligned}\nabla^{0,1}\beta &= \nabla\alpha_1 + i\nabla\alpha_2 + i(\star\nabla\alpha_1 + i \star \nabla\alpha_2) \\ &= (1 + i\star)(\nabla\alpha_1 - \star\nabla\alpha_2) \in \Omega^{0,1}(M; \mathrm{ad}(P_{\mathbb{C}}))\end{aligned}$$

which corresponds to

$$\nabla\alpha_1 - \star\nabla\alpha_2$$

Certainly $\star\nabla\star F_\nabla$ has this form! Hence it is tangent to complexified gauge orbits. Next we'll compute the Hessian of $\mathcal{YM}$ at a critical connection and show that the number of negative eigenvalues is finite.

Lemma

Let $Q\colon \Omega^1(M;\mathrm{ad}(P))\otimes\Omega^1(M;\mathrm{ad}(P))\to\mathbb{R}$ *be the Hessian of* $\mathcal{YM}$*. Then*

$$Q(a,a)=\langle\nabla^*\nabla a,a\rangle+\langle\star[\star F_\nabla,a],a\rangle$$

Proof.

We computed the variation of $\mathcal{YM}$ many aeons ago

$$F_{\nabla+ta}=F_\nabla+t\nabla a+[a,a]$$

so at a Yang-Mills connection where the first variation vanishes

$$\mathcal{YM}(\nabla+ta)=\mathcal{YM}(\nabla)+t^2\left(\langle\nabla a,\nabla a\rangle+\langle F_\nabla,[a,a]\rangle\right)+\mathcal{O}(t^3)$$

The first term of order t^2 clearly gives $\langle\nabla^*\nabla a,a\rangle$. We rearrange the second term as follows: □

Proof.

$$\int_M \mathrm{Tr}([a,a] \wedge \star F_\nabla) = \int \mathrm{Tr}(a \wedge [a, \star F_\nabla])$$
$$= \int \mathrm{Tr}(a \wedge \star(\star[\star F_\nabla, a]))$$
$$= \langle \star[\star F_\nabla, a], a\rangle$$

The first line works because we're taking the trace of the adjoint representation of a Lie algebra:

$$\mathrm{Tr}(\mathrm{ad}[X,y]\mathrm{ad}Z) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y\mathrm{ad}Z - \mathrm{ad}Y\mathrm{ad}X\mathrm{ad}Z)$$
$$= \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y\mathrm{ad}Z - \mathrm{ad}X\mathrm{ad}Z\mathrm{ad}Y)$$
$$= \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}[Y,Z])$$

since $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$. □

Now (where it's smooth) the tangent space to the space of Yang-Mills connections should be the kernel of this Hessian. This is not yet finite-dimensional, since there is still infinitesimal gauge invariance. Therefore we have

$$0 \to \Omega^0(M; \mathrm{ad}(P)) \to \ker(Q) \to T_\nabla \mathcal{M} \to 0$$

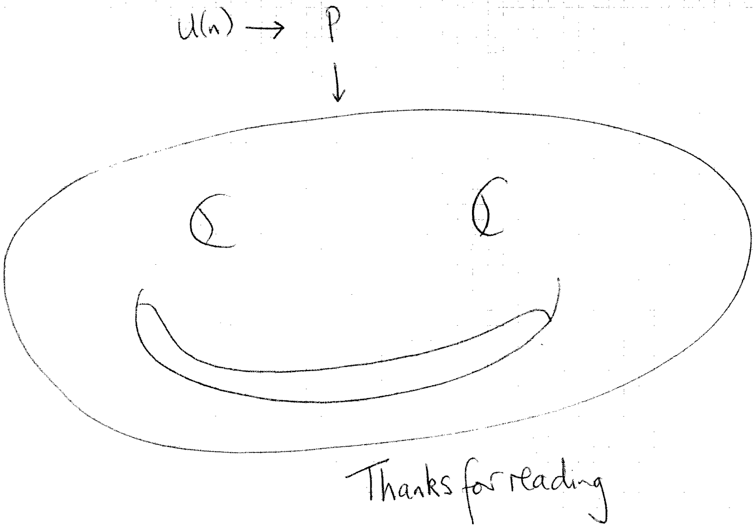
($\mathcal{M}$ is the moduli space) and the orthogonal complement of the image of $\Omega^0(M; \mathrm{ad}(P))$ under the infinitesimal gauge action $a \mapsto \nabla a$ is $\ker \nabla^*$ by Hodge theory. Hence $T_\nabla \mathcal{M}$ is the space of 1-forms with values in $\mathrm{ad}(P)$ which satisfy

$$\nabla^* \nabla a = - \star [\star F_\nabla, a] \text{ and } \nabla^* a = 0$$

The highest order part is $\nabla^* \nabla + \nabla \nabla^*$ which is elliptic and hence the tangent space is finite-dimensional. Similarly, the space of negative eigenvectors (in $\ker \nabla^*$) is finite-dimensional (the spectrum is bounded in the negative direction). Hence the Morse index is finite.

There are serious difficulties with carrying this program further (see Daskalopoulos' paper "The topology of the space of stable bundles on a compact Riemann surface" for their resolution). Atiyah-Bott elide these difficulties by avoiding the direct proof of their formula and using an algebro-geometric argument (Section 15). We have time for neither approach.

I hope that over the last twenty four lectures I've managed to communicate to you some of the beauty of Yang-Mills theory on a Riemann surface, some of the analytical techniques for dealing with it and some of the topological computations it enables one to perform. At the very least, you should be well-equipped to go and read the Atiyah-Bott paper for yourselves! The ideas we have developed here (about infinite-dimensional moment maps, about elliptic moduli problems and the Morse theoretic approach to them) have been fermenting for nearly thirty years since that paper was written and have led to some of the most powerful theorems in modern geometry. I imagine that if you continue geometry at a research level, you'll keep coming across ideas that you first read in Atiyah and Bott.


U(n) → P
Thanks for reading