Lecture 6: Principal bundles

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We are in the process of defining principal G-bundles and connections on them. The last thing we said was:

Definition

A connection is a G-equivariant choice of horizontal space $\mathcal{H}_p$ in each T_pP, i.e. a subspace which projects via π_* to $T_{\pi(p)}M$ and such that $g_*(\mathcal{H}_p) = \mathcal{H}_{g(p)}$ for all $g \in G$. We write $\tilde{X}$ for the unique horizontal vector field which projects along π_* to the vector field X on M.

We see that this is the same as a $\mathfrak{g}$-valued 1-form α on P, which projects tangent vectors onto their vertical part (i.e. $\ker \alpha = \mathcal{H}$) and takes vertical vectors $v \in \mathfrak{g}$ to themselves. G-equivariance of $\mathcal{H}$ translates into the equivariance

$$\alpha(g_* v) = g_* \alpha(v)$$

of α.

In the case of $U(1)$-bundles we saw that the difference of two connections $\mathcal{H}$ and $\mathcal{H}'$ is a 1-form on M. This has an analogue for G-bundles. Since both $\mathcal{H}_p$ and $\mathcal{H}'_p$ project bijectively to $T_{\pi(p)}M$ we can write $\mathcal{H}'_p$ as the graph of a $\mathfrak{g}$-valued matrix $A : \mathcal{H}_p \to \mathfrak{g}$. In other words we can think of $A : \pi^* TM \to P \times \mathfrak{g}$. Equivariance of $\mathcal{H}'$ now implies

$$A(g_* v) = g_* A(v) \text{ for all } g \in G$$

We want to think of A as a 1-form a living on M, so we want to divide $\pi^* TM$ and $P \times \mathfrak{g}$ by G.

Recall

Remember that on a Lie group, the tangent space at the identity e (the Lie algebra $\mathfrak{g}$) is naturally identified with the space of left-invariant vector fields ξ_g such that

$$\xi_g = (L_g)_* \xi_e$$

where L_g denotes left-multiplication by g.

The fibre of our bundle $P \times \mathfrak{g} \subset TP$ over a point $p \in P$ is precisely the space of left-invariant vertical vector fields on the fibre $P_{\pi(p)}$. There is a bundle over M whose fibre at q is the bundle of left-invariant vertical vector fields on the fibre P_q. This is clearly the bundle in which this putative 1-form a takes its values since $a(v) = A(\tilde{v})$ is a G-equivariant vertical vector field! Left-invariant is the same as G-equivariant:

$$(L_g^* A)(\tilde{v}) = A(g_* \tilde{v}) = g_* A(\tilde{v}) = (L_g)_* A(\tilde{v})$$

We want a better description of this bundle so we define the notion of *associated vector bundle*.

Associated bundles

You may be more used to/comfortable with vector bundles than principal bundles. In this case you'll be happy to know that there's a nice way to translate between the two. Given a representation V of G (if you like just bear in mind the case $G = U(n)$ acting on $\mathbb{C}^n$) and a principal G-bundle P you automatically get a vector bundle

$$P \times_G V$$

where the notation means that the total space of the bundle is $P \times V$ divided by the group action of G by

$$(p, v) \mapsto (pg, g^{-1}v)$$

This inherits a projection $\pi : P \times_G V \to M$.

Lemma

The fibre $\pi^{-1}(m)$ can be identified with the vector space V by picking $p \in P_m$ and sending

$$v \mapsto [(p, v)]$$

Proof.

Surjectivity: If $[(q, w)] \in \pi^{-1}(m)$ then $p = qg$ for some $g \in G$ so $(q, w) \sim (p, g^{-1}w)$. Injectivity: if $(p, v) \sim (p, w)$ then there is a $g \in G$ such that $p = pg$, $v = g^{-1}w$, however the G-action on P is free so $g = 1$. □

Lemma

Sections of the bundle associated to P via a representation ρ correspond 1-1 with maps $f : P \to V$ satisfying

$$f(pg) = \rho(g^{-1})f(p)$$

Proof.

Given such a map, form the map $F = \mathrm{id} \times f : P \to P \times V$. This is G-equivariant with respect to the usual G-action P and the diagonal G-action on $P \times V$ and hence it descends to a section $M \to P \times_G V$. Conversely a section σ lifts to a G-equivariant map $F : P \to P \times V$ and since $\pi \circ \sigma = \mathrm{id}$, the map F has the form $\mathrm{id} \times f$. □

One can recover P as the bundle of frames in the vector bundle $P \times_G V$. A frame is just a G-equivariant identification of the fibre with V (i.e. a choice of basis! The basis is orthonormal, unitary or whatever else G may be) which is precisely what we picked to see that the fibre of $P \times_G V$ over m was isomorphic to V.

Lemma

The bundle over M we described earlier, whose fibre over $q \in M$ is the space of left-invariant vertical vector fields on P_q, is associated to P via the adjoint action of G on $\mathfrak{g}$

Recall that the adjoint action of G on itself is $\mathrm{Ad}_g(h) = ghg^{-1}$ and the adjoint action on the Lie algebra is $\mathrm{ad}_g = d\mathrm{Ad}_g = dR_{g^{-1}} \circ dL_g$ where R_g and L_g are the right and left actions of G on itself.

Proof.

To each $\xi \in \mathfrak{g}$ associate the vector field $V(\xi)_p = \frac{d}{dt}\big|_{t=0}(pe^{t\xi})$. If we write R_g for the right action of G on itself then

$$\begin{aligned}(R_g)_* V(\xi)_p &= \frac{d}{dt}\bigg|_{t=0}\left(pe^{t\xi}g\right)\\ &= \frac{d}{dt}\bigg|_{t=0}\left(pg\left(g^{-1}e^{t\xi}g\right)\right)\\ &= V(\mathrm{ad}_{g^{-1}}\xi)_{pg}.\end{aligned}$$

Note that a section of the bundle associated to P via the representation $\rho : G \to \mathrm{Aut}(V)$ lifts to a map $f : P \to V$ satisfying

$$R_g^* f = \rho(g^{-1})f.$$

□

Outcome

The difference of two connections on P is an $\mathrm{ad}(P)$-valued 1-form on M, where $\mathrm{ad}(P)$ is the vector bundle associated to P via the representation $\mathrm{ad} : G \to \mathrm{Aut}(\mathfrak{g})$. If we fix a connection ∇ then for any $\mathrm{ad}(P)$-valued 1-form A we write $\nabla_A = \nabla + A$ and we observe that the space of connections $\mathcal{A}$ is an affine space modelled on the space of $\mathrm{ad}(P)$-valued 1-forms. In particular it is contractible!

Differentiating sections of associated bundles

A connection on P induces a connection on the vector bundle E associated to some representation $\rho : G \to \mathrm{Aut}(V)$. Recall that a connection on a vector bundle is just a differential operator on sections σ which obeys the Leibniz rule with respect to functions f:

$$\nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma$$

Sections of the associated bundle are in 1-1 correspondence with G-equivariant maps $\sigma : P \to V$ and we can define

$$\nabla_X\sigma = d\sigma(\tilde{X}) = \mathrm{Lie}_{\tilde{X}}\sigma$$

(The last equality holds by Cartan's formula, considering σ as a "V-tuple" of functions).

A step back

What I have written is actually extremely confusing (for a change...). Let's deconstruct it in the case of the trivial $U(1)$-bundle and the associated complex vector bundle E (with the usual action of $U(1)$ on $\mathbb{C}$). Here $P = M \times U(1)$, $E = M \times \mathbb{C}$ and a section is the same as a map $s : M \to \mathbb{C}$. The corresponding map $\sigma : P \to \mathbb{C}$ is NOT just $s \circ \pi$, rather it is

$$\sigma(m, e^{i\theta}) \mapsto e^{-i\theta} s(m)$$

Now if $\nabla = d$ is the trivial connection, $\tilde{X} = (X, 0)$ so

$$\nabla_X \sigma = e^{-i\theta} ds(X)$$

If $\nabla = d + iA$ then $\tilde{X} = (X, -A(X))$ so

$$\nabla_X \sigma = e^{-i\theta}(ds(X) + iA(X)s(m))$$

The reason things are complicated is because we are remembering more information than we need (a whole G's worth of frame data) in order to make it easier to write down.