Lecture 8: The Kempf-Ness theorem

Jonathan Evans

13th October 2011

We are interested in the action of $\mathcal{G}$ on the infinite-dimensional affine space $\mathcal{A}$. To get a picture of what to expect we study a finite-dimensional analogue:

- let V be a complex vector space with a Hermitian inner product (write $||\cdot||$ for the corresponding norm),
- let $S^1 \to U(V)$ be an action of the circle by unitary matrices,
- let $\mathbb{C}^* \to GL(V)$ be the complexification of this action.

We want to understand the quotient space $V/\mathbb{C}^*$ but this can be quite unpleasant. Next time we will complexify the action of $\mathcal{G}$ on $\mathcal{A}$ and see what this has to do with anything.

Example

Consider $\lambda \in \mathbb{C}^*$ acting on $\mathbb{C}^2$ by $(x, y) \mapsto (\lambda^{-1}x, \lambda y)$. The orbits are:

- the conics $xy = c \neq 0 \in \mathbb{C}$,
- the axes $y = 0, x \neq 0$, $x = 0, y \neq 0$,
- the origin.

Since the axes come arbitrarily close to 0 it is clear that the quotient topology on the orbit space is non-Hausdorff. However, $\mathbb{C}^2 \setminus \{\text{axes}\}/\mathbb{C}^*$ is Hausdorff, in fact it's homeomorphic to $\mathbb{C}$.

More generally we want to form Hausdorff quotients by considering only the orbits which are closed sets.

Definition

A point $v \in V$ is stable if its orbit under $\mathbb{C}^*$ is closed.

Theorem (Kempf-Ness)

A point v is stable if and only if the function $||\cdot||^2$ restricted to its orbit attains its minimum.

We can think of this function as a function on $p_v : \mathbb{C}^* \to \mathbb{R}$, given by $p_v(g) = ||g(v)||^2$. Note that since the norm is $U(V)$-invariant the function p_v is S^1-invariant and descends to a function on $(\mathbb{C}^*/S^1, \times) \overset{\log}{\to} (\mathbb{R}, +)$

$$p_v(x) = ||e^x(v)||^2$$

In our example above $e^x(v_1, v_2) = (e^{-x}v_1, e^x v_2)$ so $p_v(x) = ||v_1||^2 e^{-2x} + ||v_2||^2 e^{2x}$. We see that this has a minimum at

$$\frac{1}{2}\left(\log(||v_1||) - \log(||v_2||)\right)$$

if both v_1 and v_2 are nonzero, at 0 if $v = 0$ and the minimum is not attained along the two punctured axes. In fact this example is representative.

The S^1-action is reducible and so V splits as an orthogonal direct sum $V_1 \oplus \cdots \oplus V_n$ of 1-dimensional representations where S^1 acts on V_m as $v_m \mapsto \lambda^{j_m} v_m$ for some weight j_m (in our example the weights were $-1, 1$). Therefore $p_v(x) = \sum_m ||v_m||^2 e^{2j_m x} = \sum_{k=-\infty}^{\infty} a_k e^{kx}$ (where only finitely many coefficients are nonzero). We divide our analysis into three cases

- Type I: $a_k = 0$ for all $k \neq 0$. In this case the minimum is obviously attained and the orbit is obviously closed since $j_m = 0$ so the action fixes v.
- Type II: $a_k = 0$ for all $k < 0$ (resp. $k > 0$) and $a_k \neq 0$ for some $k > 0$ (resp. $k < 0$). In this case the minimum is obviously not attained and the orbit is obviously not closed since $e^x(v)$ tends to an orbit of the first type as $x \to -\infty$ (resp. ∞).
- Type III: there is a $k > 0$ and a $k' < 0$ such that $a_k \neq 0$ and $a_{k'} \neq 0$. In this case the minimum is obviously attained (just do the calculus). We will now show that this implies v is stable.

Lemma

If v is not stable then p_v does not attain its minimum.

Proof.

If v is not stable then its orbit is not closed so there exists $w \in V$ such that $w \in \overline{\mathbb{C}^*(v)}$ but $w \notin \mathbb{C}^*(v)$, so either $w = \lim_{x \to \pm\infty} e^x(v)$. The corresponding limit $\lim_{x \to \pm\infty} p_v(x) = p_v(w)$ is finite and hence the j_m are either all nonpositive or all nonnegative. Since $w \neq v$ there must be one j_m which is nonzero. It's now easy to see that the function $p_v(x)$ is of type II and hence does not attain its minimum. □

This completes our proof of the Kempf-Ness theorem.

To understand the space of stable points it's therefore important to understand the critical points of p_v and what better way than by differentiating it?

$$\frac{dp_v}{dx} = 2\sum_{m=1}^{n} j_m||v_m||^2 e^{2j_m x}$$

Suppose that v is stable and that the minimum occurs at $x = x_0$. WLOG $x_0 = 0$ because we can always replace v by $e^{-x_0}v$. Therefore the orbit of a stable vector contains a zero of the function

$$\mu = \sum_{m=1}^{n} j_m||v_m||^2 : V \to \mathbb{R}$$

In fact it contains a whole S^1 of such zeros since μ is S^1-invariant.

Theorem

Let V^s denote the space of stable vectors under the action of $\mathbb{C}^*$. Then

$$V^s/\mathbb{C}^* = \mu^{-1}(0)/S^1.$$

Let V be a vector space and $Q : V \otimes V \to \mathbb{R}$ a nondegenerate bilinear form. Then we can translate 1-forms χ into vector fields X by defining

$$\chi(Y) = Q(X, Y) \text{ for all } Y \in TV$$

In particular if $f : V \to \mathbb{R}$ is a function then $df \in \Omega^1(V)$ is a 1-form and it yields a vector field $\mathrm{Qgrad}(f)$ by $df(Y) = Q(\mathrm{Qgrad}(f), Y)$.

- If Q is positive definite and symmetric this gives exactly the gradient of f and
$$\mathrm{Lie}_{\mathrm{Qgrad}(f)} f = |\mathrm{Qgrad}(f)|^2 > 0$$
- If Q is instead antisymmetric then $\mathrm{Qgrad}(f)$ behaves very differently and
$$\mathrm{Lie}_{\mathrm{Qgrad}(f)} f = 0$$
so f is preserved by this flow.

For example take the function $f(x, y) = x^2 + y^2$ on $\mathbb{R}^2$ and take $Q = dx \wedge dy$. Then

$$\mathrm{Qgrad}(f) = -y\partial_x + x\partial_y$$

is the vector field generating a rotation around the origin. This preserves the level sets (constant radius). In general this procedure works very well on *symplectic manifolds* (Q a nondegenerate, alternating, *closed* 2-form) since there

$$\mathrm{Lie}_{\mathrm{Qgrad}(f)} Q = d\iota_{\mathrm{Qgrad}(f)} Q = ddf = 0$$

We call f the *Hamiltonian* generating $\mathrm{Qgrad}(f)$ and observe that the Hamiltonian is preserved by the Hamiltonian flow.

Now return to our example where $V_m = \mathbb{C}$ is a 1-dimensional representation of S^1 with weight j_m. This is generated by a vector field which is just generated by the Hamiltonian $j_m \mathrm{Qgrad}(||v_m||^2)$ (as in the previous slide) where

$$Q = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

is the imaginary part of the Hermitian structure on V. This is precisely the summand of μ corresponding to this summand of V. Adding up the Hamiltonians on $V = \bigoplus_{m=1}^n V_m$ gives μ and this Hamiltonian generates precisely our original action of S^1 on V. We call μ a moment map for the circle action.

Next time we will see a moment map for the action of $\mathcal{G}$ on $\mathcal{A}$, we will complexify the action of $\mathcal{G}$ and we will state a theorem analogous to the Kempf-Ness theorem in this infinite-dimensional setting. This is the Narasimhan-Seshadri theorem.