Workshop 5

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This is modified from an old MATH105 exam. Now we only have one MATH105 question on the MATH100 paper, so this is worth five times the practice it used to be!

1. Here are some matrices:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In each case, state which of the matrices (if any):

(a) have image inside the xz-plane.

(b) represent a rotation of some kind.

(c) permute the x, y and z axes in some way.

(d) satisfy $M^n = 0$ for some n

(e) have 0 as an eigenvalue.

(f) have determinant -1.

2. Consider the system of simultaneous equations $x+y=1$, $x-y+1=0$ and $y=1-x/2$.

   (a) Sketch the three lines in $\mathbb{R}^2$ defined by these three equations.

   (b) Does this system of equations have a solution? If so, what is it? How can you see this in your sketch?

   (c) Write down the 3-by-3 augmented matrix M for this system of equations.

   (d) If you were to put M into reduced echelon form, the final row would be zero: with no further calculation, explain why this follows from your answer to (b).

   (d) Put M into reduced echelon form, and check that the final row is indeed zero.

3. Let

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

   (a) Find the determinant of M by expanding around a row or column.

   (b) Find the inverse of M by putting the 4-by-8 augmented matrix into reduced echelon form.

   (c) How many row operations did you perform in part (b)? Explain why this gives an upper bound on the number n of elementary matrices E_i required to express M as a product $E_1 \cdots E_n$.

4. For each of the following statements, state if it is true or false. If it is true, give a proof; if it is false, give a counterexample.

   (a) If A and B are 2-by-2 matrices such that $AB=0$ then neither A nor B is invertible.

   (b) If A is a 3-by-3 matrix whose image is a 2-dimensional plane then $\det(A)=0$.

   (c) There is no real 2-by-2 matrix A such that $A^2=-I$.

   (d) The characteristic polynomial of an n-by-n matrix has degree n.

5. Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) A corresponds to a rotation in 3d. What is the axis fixed by this rotation?

(c) Check that the vector $v := \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$ is orthogonal to the axis of rotation.

(d) Calculate the angle between v and Av.

(e) Describe the action of A on the three vectors $e_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Write down the inverse of A without doing any further computation.