Linear Algebra: Moodle questions

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1 Week 1

1.1 Questions

Question 1.1. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Each sentence below can be written succinctly as an equation of 2-by-2 matrix products. Match the equation to the sentence.

Sentences:

(a) If you project orthogonally to the x-axis and then to the y-axis, everything goes to the origin.

(b) If you project orthogonally to the y-axis and then to the x-axis, everything goes to the origin.

(c) If you project to the x-axis and then rotate by 90 degrees anticlockwise, that's the same as rotating by 90 degrees anticlockwise and then projecting to the y-axis.

Equations:

(i) $BC = 0$

(ii) $CB = 0$

(iii) $ABCA = I$

(iv) $AB = CA$

(v) $ABCA = 0$.

(vi) $AC = BA$

Question 1.2. Match the matrix to its action on $\mathbb{R}^2$.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}$$

(i) Stretching the x-direction by a factor of 2.

(ii) Stretching the y-direction by a factor of a half.

(iii) Rotating by 90 degrees clockwise around the origin.

(iv) Rotating by 90 degrees anticlockwise around the origin.

(v) Rotating by 90 degrees clockwise around (1,1).

(vi) Reflecting in the x-axis.

(vii) Reflecting in the line x = y.

(viii) Reflecting in the line x = -y.

(ix) The identity transformation.

(x) None of the other options.

Question 1.3. In this question, s and t are variables on which our vectors and matrices depend. For which vector $v(s,t)$ do we have $\begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \end{pmatrix} v(s,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all s, t?

(a) $\begin{pmatrix} -s \\ -t \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ s+t \end{pmatrix}$

(c) $\begin{pmatrix} -s \\ -t \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 1 \\ s+t \end{pmatrix}$

Question 1.4. Recall that $A(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is the matrix for a rotation by an angle θ around the origin and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix for a reflection in the y-axis. By multiplying out a suitable combination of matrices (or otherwise), determine whether the following statement is true or false. "Reflecting in the y-axis, then rotating by θ around the origin, then reflecting back again is the same as rotating by $-\theta$ around the origin."

2 Week 2

2.1 Questions

Question 2.1. For each matrix, decide whether it (a) is in reduced echelon form for all values of c, (b) is not in reduced echelon form for any value of c, (c) is in reduced echelon form for some values of c.

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & c \end{pmatrix}, \quad Q = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 2.2. Suppose that $t \neq 1$ is a number. Find the reduced echelon form of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & t \end{pmatrix}$. How many leading entries does it have?

Question 2.3. For each matrix below, consider the system of simultaneous equations $Mv = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. For which matrix M does this system have a solution?

(a) $\begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Question 2.4. Which of the following 4-by-4 matrices is not orthogonal?

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3 Week 3

Question 3.1. Let's think of the set of n-by-n matrices as $\mathbb{R}^{n^2}$ by using the matrix entries as coordinates. Let $D \subset \mathbb{R}^{n^2}$ be the subset of matrices with determinant zero. Select all the statements which are true.

(a) The subset D is closed under rescaling.

(b) The subset D is closed under addition.

(c) The subset D contains the origin.

(d) The subset D is an affine subspace.

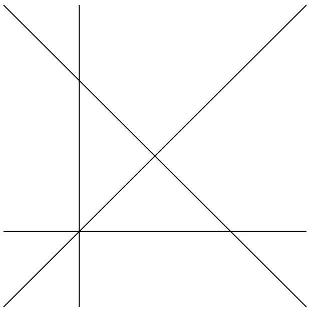
Question 3.2. The diagram below shows a collection of lines which represent a system of simultaneous linear equations in two variables. How many solutions does this system have?

(a) None.

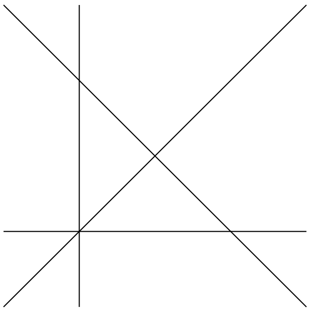
(b) One.

(c) Four.

(d) Infinitely many.



Question 3.3. Here we see the same configuration of lines as in the previous question. One of the following systems of simultaneous equations is being depicted; which one?



(a)
$$x = 0,\ y = 0,\ x + y = 0,\ x - y = 1$$

(b)
$$x = 1,\ y = 0,\ x + y = 1,\ x - y = 1$$

(c)
$$x = 0,\ y = 0,\ x + y = 1,\ x - y = 0$$

(d)
$$x = 0,\ y = 0,\ x + y = 0,\ x - y = 0$$

Question 3.4. Select all the matrices whose determinant is zero. (Hint: You shouldn't have to do much work to compute each one.)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix},\ F = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \\ 2 & 3 & 5 & 8 & 13 \\ 3 & 5 & 8 & 13 & 21 \\ 5 & 8 & 13 & 21 & 36 \end{pmatrix}$$

4 Week 4

Question 4.1. Consider the matrix $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & c \end{pmatrix}$. For which value of c does M fail to be invertible?

Question 4.2. Select all true statements:

- For all a, b, c, d, the number a is an eigenvalue of $\begin{pmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \end{pmatrix}$.
- For all a, b, c, d, the number $-a$ is an eigenvalue of $\begin{pmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \end{pmatrix}$.
- For all a, b, c, d, the number 0 is an eigenvalue of $\begin{pmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \end{pmatrix}$.
- If M is a real matrix and λ is a real eigenvalue then there is a nonzero real eigenvector v.
- If M is an upper triangular matrix with integer entries then its eigenvalues are integers.

Question 4.3. You are trying to find the eigenvalues of a particularly troublesome 5-by-5 matrix. You have found the characteristic polynomial, but cannot figure out how to find its roots. You think the lecturer must have made a mistake: it seems like there's no way of solving this polynomial. In the end, you type the polynomial into the online calculator Lupusaries A and it gives you the five distinct roots correct to 12 decimal places. To get 200 more decimal places, you'd need a paid subscription. Being a cheapskate, you press on and solve the eigenvector equation $Mv = \lambda v$ with the various approximate eigenvalues λ given to you by Lupusaries A. What do you find?

- You find a nonzero eigenvector correct to 12 decimal places.
- You find a line of eigenvectors in 5-d whose slopes (ratios of pairs of coordinates) are correct to 12 decimal places.
- You find that $v = 0$ is the only solution.
- You find eigenvectors, but they are very far from the correct solution, and you could only have gotten something good to 12 decimal places by getting a subscription.
- None of the above.