Linear Algebra: Practice questions

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1 Week 1

1.1 Questions

Exercise 1.1. In each case, compute the answer if it is well-defined; if it is not well-defined, say so.

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 7 & -2 & 3 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 17 \\ -83 & 21.5 \\ 92 & 43 \end{pmatrix}\begin{pmatrix} 1 \\ 45 \\ 12 \end{pmatrix}.$$

Exercise 1.2. Compute $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Exercise 1.3. Consider the matrices below. Precisely one of AB, BA, AC, CA, BC and CB is well-defined. Which one, and what is the result of the multiplication?

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 7 & -1 \\ 8 & 5 & 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 \\ 8 \\ -1 \\ 2 \\ -3 \end{pmatrix}$$

Exercise 1.4. For which of the following matrices P does there exist a positive integer n such that $P^n = I$? Justify your answers

- A, the matrix for a rotation of the plane by an angle $2\pi q$ radians, where q is a rational number.
- B, the matrix for a rotation of the plane by an angle $2\pi q$ radians, where q is an irrational number.
- C, the matrix for a reflection of $\mathbb{R}^2$ in a line through the origin.

Exercise 1.5. Write down:

(a) The matrix F such that Fv is the vertical projection of v to the x-axis.

(b) The matrix G such that Gv is the rotation of v by 90 degrees clockwise.

Do any of the following matrix products yield the zero matrix? If so, which ones? Justify your answers. (Hint: You should be able to answer this without multiplying any matrices together, just by thinking about what F and G do.)

(i) GFG (i) GFF (i) FGF

Exercise 1.6. Consider the matrices

$$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

For each matrix A above:

(a) Find a vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $Av = v$. *Hint: this amounts to solving some simultaneous equations.*

(b) Sketch the vector v you found in part (a). Sketch the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and Ae_1.

(c) Can you describe the action of A geometrically?

Exercise 1.7. Let $M = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$. The transformation $v \mapsto Mv$ is a rotation of 3-dimensional space. Calculate the axis and angle of rotation.

Exercise 1.8. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

(a) Find a vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $Av = v$.

(b) Sketch the vectors v, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Can you describe the action of A geometrically?

Exercise 1.9. In each case, determine whether the statement is true or false, and give a proof or counterexample accordingly.

(a) If A is a nonzero matrix then A^n is nonzero for all positive integers n. *Here A^n means A multiplied with itself n times, e.g. $A^2 = AA$.*

(b) Suppose that A and B are matrices such that $AB = I$, and suppose that v is a vector such that $Bv = 0$. Then $v = 0$.

Exercise 1.10. Encode a cubic polynomial $P = a_3x^3 + a_2x^2 + a_1x + a_0$ by its vector of coefficients $v_P = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

(a) What is the vector of coefficients of dP/dx?

(b) Write down a 4-by-4 matrix ∂ such that $\partial v_P = v_{dP/dx}$.

(c) What polynomial corresponds to $\partial^2 v_P$?

1.2 Questions for enthusiasts

Exercise 1.11. Consider the matrix $H_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$. Show that $H_\phi^2 = I$. What kind of geometric transformation does the matrix $H_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$ represent? (Hint: Find a vector v such that $H_\phi v = v$; remember your half-angle trigonometric formulae!)

Exercise 1.12. Let M be a square matrix. We define the exponential of M to be the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{n!} M^n = I + M + \frac{1}{2} M^2 + \frac{1}{3!} M^3 + \cdots . \tag{1}$$

Here M^n is defined recursively for $n \geq 0$ by $M^0 = I$, $M^{n+1} = M(M^n)$.

Compute $\exp(M)$ for the following matrices (t is just a variable, $i = \sqrt{-1}$):

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & ti \\ ti & 0 \end{pmatrix}.$$

Hopefully, in this exercise, you found that $\exp(D)$ *is a rotation matrix. Something similar holds in higher dimensions: rotations are precisely the exponentials of* antisymmetric matrices, *i.e. matrices M such that $M^T = -M$.*

Exercise 1.13 (Special relativity velocity addition)**.** Given a number v, define the matrix $\Lambda(v) = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix}$. Check that

$$\frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} = \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}}$$

for all u, v. Deduce that

$$\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right).$$

2 Week 2

Exercise 2.1. Let

$$C = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Which of these matrices are in echelon form?

(b) Which of these matrices are in reduced echelon form?

(c) For each matrix in reduced echelon form, state how many free variables there are.

Exercise 2.2. Consider this system of equations

$$\begin{aligned} x - y + z &= -1 \\ 2x + z &= 5 \\ 3x + y - z &= 5 \end{aligned}$$

(a) Write it as a matrix equation and write the augmented matrix.

(b) Put the augmented matrix into reduced echelon form using row operations, and hence give the general solution of the system of equations.

Exercise 2.3. For which value of the constant c does the following system of simultaneous equations have no solution? Explain your answer.

$$\begin{aligned} x + y - cz &= 1, \\ z + y &= c, \\ x - y &= 2. \end{aligned}$$

Exercise 2.4. Put the matrix $\begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$ into echelon form. How many leading entries does it have?

Exercise 2.5. Find the axis and angle of rotation for this 3-dimensional rotation matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

(Hint: The vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ *should be orthogonal to the axis.)*

Exercise 2.6. Suppose that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is an orthogonal matrix. What are the possible values of a, b, c?

Suppose that $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ is an orthogonal matrix. What can you say about a?

Exercise 2.7. *We say that a square matrix A is symmetric if $A^T = A$ and antisymmetric if $A^T = -A$.* Are the following statements true or false?

(a) For any square matrix M, $M + M^T$ is symmetric.

(b) For any square matrix M, $M - M^T$ is antisymmetric.

(c) Any square matrix is the sum of a symmetric matrix and an antisymmetric matrix.

2.1 Questions for enthusiasts

Exercise 2.8. Suppose that A is an n-by-n matrix whose columns are the vectors $v_1, \ldots, v_n$. Show that A is an orthogonal matrix (i.e. $A^T A = I$) if and only if

$$v_i \cdot v_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for all } i, j.$$

In other words, the columns of A are unit vectors which are orthogonal to one another (this is where the name "orthogonal matrix" comes from). *Hint: Write out the ijth entry of $A^T A$ using index notation.*

Exercise 2.9. Recall the definition of matrix exponentiation: $\exp(A) = I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$.

(a) By working with the power series, show that:

(i) $\exp(A^T) = (\exp(A))^T$,

(ii) $\frac{d}{dt}(\exp(tA)) = A\exp(tA)$ (here and below, t is just a real number).

(iii) $A\exp(tA) = \exp(tA)A$

(a) Suppose that $\exp(tA)$ is an orthogonal matrix for all $t \in \mathbb{R}$. By differentiating the equation $\exp(tA)^T \exp(tA) = I$ with respect to t, show that A is antisymmetric. *Hint: You may assume Leibniz's rule holds for differentiating products of matrices.*

3 Week 3

3.1 Questions

Exercise 3.1. Find the inverse of the matrices below:

$$A = \begin{pmatrix} 1 & 2 \\ 7 & 13 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

Exercise 3.2. Find the determinant of $\begin{pmatrix} 0 & -2 & 0 & 1 \\ 0 & -2 & -3 & 3 \\ 1 & 4 & 0 & -1 \\ 1 & 2 & -3 & 0 \end{pmatrix}$.

Exercise 3.3. Find the determinant and inverse of this matrix by using row operations and echelon form:

$$\begin{pmatrix} 1 & 1 & -3 & 1 \\ -1 & 2 & 0 & 1 \\ 0 & 3 & -3 & 1 \\ -1 & 2 & 3 & 0 \end{pmatrix}.$$

Exercise 3.4. Give counterexamples to the following statements:

(a) If A is a 2-by-3 matrix then there is no matrix B such that $AB = I$.

(b) It is impossible for four distinct planes in $\mathbb{R}^3$ to intersect along a line.

(c) It is impossible for four simultaneous linear equations in three variables to have a one-dimensional space of solutions.

Exercise 3.5. Prove that if $V \subset \mathbb{R}^n$ is a linear subspace and $v \in V$ then $v + V = V$ (Hint: show that $V \subset v + V$ and $v + V \subset V$.

Exercise 3.6. Which description best fits the space of solutions of the system

$$\begin{aligned} x + 2y - z &= 0 \\ x + y &= 1 \\ z - y &= 1. \end{aligned}$$

(a) a point in $\mathbb{R}^3$,

(b) a 2-dimensional affine subspace of $\mathbb{R}^3$,

(c) a 1-dimensional linear subspace of $\mathbb{R}^3$,

(d) a 1-dimensional affine subspace of $\mathbb{R}^3$,

(e) a hyperplane.

3.2 Questions for enthusiasts

Exercise 3.7. The *Jacobian* of a differentiable map $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is the m-by-n matrix

$$Jac(F) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_{n-1}} & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_{n-1}} & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_{n-1}} & \frac{\partial F_m}{\partial x_n} \end{pmatrix},$$

where $F(x_1, \ldots, x_n) = \begin{pmatrix} F_1(x_1, \ldots, x_n) \\ \vdots \\ F_m(x_1, \ldots, x_n) \end{pmatrix}$.

Find $\det(Jac(F))$ in the following examples:

1. $m = n = 2$, $F(r, \phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$.
2. $m = n = 3$, $F(r, \theta, \phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$

If $m = n$ and F is a change of coordinates $\mathbb{R}^n \to \mathbb{R}^n$ then the determinant of the Jacobian matrix is an important quantity: if $y = F(x)$ then the volume element $dy_1 \cdots dy_n$ is equal to $\det(Jac(F))dx_1 \cdots dx_n$. In the examples we've just computed the volume element in polar and spherical coordinates.

Exercise 3.8. For this question, you will need the fact (proved in week 4) that $\det(MN) = \det(M)\det(N)$. Suppose that A is an n-by-n orthogonal matrix ($AA^T = I$).

1. Show that $\det(A) = \pm 1$.
2. If $\det(A) = 1$, show that $\det(A - I) = (-1)^n \det(A - I)$. (Hint: Use the fact that $A - I = A(I - A^T)$.)
3. Deduce that if n is odd then any orthogonal matrix with determinant one has a fixed vector.

4 Week 4

Exercise 4.1. By expanding about a suitably chosen row or column, find the determinant of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}.$$

Exercise 4.2. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

Exercise 4.3. What are the eigenvalues of the matrix $\begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$?

Exercise 4.4. Let c be a variable. Find the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 2 & -1 & c \\ -1 & 2 & -1 \\ c & -1 & 2 \end{pmatrix}.$$

For which values of c does M have 0 as an eigenvalue? In each such case, find the 0-eigenvector.

Exercise 4.5. True or false? In each case, either give a proof or a counterexample.

(a) If A is a lower triangular matrix then its diagonal entries are also its eigenvalues.

(b) Suppose λ and μ are numbers such that $\lambda \neq \mu$. If v is a λ-eigenvector of A and w is a μ-eigenvector of A then $v + w$ is an eigenvector of A.

(c) The eigenspace of $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with eigenvalue 1 is 1-dimensional.

Exercise 4.6. True or false? In each case, give a proof or a counterexample.

(a) If X is a matrix in reduced echelon form then X is invertible.

(b) If X is a square matrix such that $X^2 = -I$ then X is invertible.

(c) Any matrix has at least one (possibly complex) nonzero eigenvector.

(d) If X and Y are both 2-by-2 matrices with determinant zero then $X + Y$ is not invertible.

(e) If X is invertible then all of its eigenvalues are nonzero.

Exercise 4.7. Which of the following vectors is an eigenvector of $\begin{pmatrix} 1 & 0 & t \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$?

(a) $\begin{pmatrix} 1 \\ 0 \\ -t \end{pmatrix}$, (b) $\begin{pmatrix} t \\ 0 \\ 2 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Exercise 4.8. Calculate the determinants of the following matrices (they shouldn't be as bad as they first look):

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

4.1 Questions for enthusiasts

Exercise 4.9. As in Week 1, let's encode a cubic polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ by its vector of coefficients $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

(a) What matrix corresponds to the map $P \mapsto \frac{d^2P}{dx^2} - x\frac{dP}{dx}$?

(b) What are the eigenvalues of this matrix?
[Hint: It should be upper triangular, which makes the characteristic polynomial easier to compute.]

(c) What are the eigenvectors of this matrix?

(d) For each eigenvalue λ, write down a polynomial P_λ which has a λ-eigenvector as its vectors of coefficients.

(e) For each λ, check that P_λ satisfies the differential equation

$$\frac{d^2P_\lambda}{dx^2} - x\frac{dP_\lambda}{dx} = \lambda P_\lambda.$$

Exercise 4.10. Suppose that A is a symmetric matrix (i.e. $A^T = A$). Suppose that v, w are eigenvectors of A with eigenvalues λ, μ respectively. By considering the expression

$$w^T A v,$$

show that either $\lambda = \mu$ or $w \cdot v = 0$. *[Hint: Remember that $w^T A = (A^T w)^T$. Try evaluating $w^T A v$ in two different ways and comparing the answers.]*

5 Week 5

5.1 Questions

Exercise 5.1. Consider the ellipsoid defined by the equation

$$2(x^2 + y^2 + z^2 - xy - yz) = 1.$$

Write this equation in the form $v^T A v = 1$ for $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and some matrix A. Find the eigenvalues of A and check they are all positive. Find the principal directions and principal radii of this ellipsoid. (Hint: Recall from lectures that if the ellipsoid is cut out by the equation $v^T A v = 1$ then the principal directions are the eigendirections for A and the principal radii are $1/\sqrt{\lambda}$ for the corresponding eigenvalues λ.)

Exercise 5.2. Let a, b be two numbers and consider the matrix $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. What are the eigenvalues and eigenvectors of this matrix? Suppose that $a+b > 1$ and $1 > a-b > 0$. What happens to the vectors $M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $n \to \infty$?

Exercise 5.3. Find the kernel and nullity for each of the following matrices

$$A = \begin{pmatrix} 1 & 0 & 7 \\ 4 & 2 & 1 \\ 3 & 2 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}.$$

What is the rank in each case?

Exercise 5.4. We say that a complex matrix A is *Hermitian* if $\bar{A}^T = A$ (here the bar means complex conjugation of each matrix entry). Show that any eigenvalue of a Hermitian matrix is real and that if v and w are eigenvectors for distinct eigenvalues $\lambda \neq \mu$ then $\bar{v}^T w = 0$. *These observations are important in quantum mechanics, because eigenvalues of Hermitian operators are what we measure as observable quantities like energy.*

Exercise 5.5. Find the limit $\lim_{n\to\infty} \frac{P_{n+1}}{P_n}$ where P_n is the sequence of Pell numbers (defined by the recurrence $P_{n+2} = 2P_{n+1} + P_n$, $P_1 = 0$, $P_2 = 1$).

Exercise 5.6. Find the characteristic polynomials of the matrices:

$$\begin{pmatrix} 0 & -c_0 \\ 1 & -c_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -c_0 \\ 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \end{pmatrix}.$$

Have a guess at the characteristic polynomial of

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & 0 & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} \end{pmatrix}.$$

5.2 Questions for enthusiasts

Exercise 5.7. Prove your guess for Exercise 5.6 by induction. Solve Exercise 5.6 using as many different methods to compute determinants as you can.

Exercise 5.8. Given numbers $\lambda_1, \ldots, \lambda_n$, the matrix

$$V = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is called the *Vandermonde matrix.* Consider the quantity

$$Q = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

(for example, if $n = 3$, $Q = (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$). Verify that $\det(V) = Q$ for $n = 2$ and $n = 3$. Show that both $\det(V)$ and Q vanish if $\lambda_k = \lambda_\ell$ for some $k \neq \ell$.

In fact, $\det(V) = Q$ *for all* n.

Exercise 5.9. Let $C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & 0 & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} \end{pmatrix}$, (see Exercise 5.6 for its characteristic polynomial!). Show that if λ is an eigenvalue of C then the row vector $w = \begin{pmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-1} \end{pmatrix}$ satisfies $wC = \lambda w$. Deduce that if C has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and V is the Vandermonde matrix from Exercise 5.8 whose ith row is $\begin{pmatrix} 1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} \end{pmatrix}$ then VCV^{-1} is the diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Hence prove that $Tr(C^m) = \sum_{i=1}^n \lambda_i^m$ for all m.

Exercise 5.10. If $t^n + c_{n-1}t^{n-1} + \cdots + c_0$ is a polynomial of degree n with leading coefficient 1 and distinct roots, show that the sum of squares of its roots is $c_{n-1}^2 - 2c_{n-2}$ and find a similar expression for the sum of cubes of the roots. (Hint: Use Exercise 5.9)

Exercise 5.11. Given a polynomial $p(z) = p_m z^m + \cdots + p_0$ of degree m and a polynomial $q(z) = q_n z^n + \cdots + q_0$ of degree n, the *resultant* of p and q is defined to be the determinant of the $(m+n)$-by-$(m+n)$ matrix

$$\begin{pmatrix}
p_m & p_{m-1} & \cdots & p_1 & p_0 & 0 & 0 & \cdots & 0 \\
0 & p_m & p_{m-1} & \cdots & p_1 & p_0 & 0 & \ddots & \vdots \\
0 & 0 & p_m & p_{m-1} & \cdots & p_1 & p_0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & p_m & p_{m-1} & \cdots & p_1 & p_0 \\
q_n & q_{n-1} & \cdots & q_0 & 0 & 0 & 0 & \cdots & 0 \\
0 & q_m & q_{m-1} & \cdots & q_0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & q_m & q_{m-1} & \cdots & q_0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & \vdots \\
\vdots & & & & & \ddots & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & q_m & q_{m-1} & \cdots & q_0
\end{pmatrix}$$

It is a theorem of Sylvester that the resultant of p and q vanishes if and only if p and q have a common factor.

Write this matrix out in the case when $m = 2$, $n = 1$ and verify that the resultant vanishes if and only if there exists a polynomial $a + bz$ such that $p(z) = q(z)(a + bz)$).

1 Week 1

1.1 Solutions

Solution 1.1. (a) $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & 3 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 17 \\ -83 & 21.5 \\ 92 & 43 \end{pmatrix} \begin{pmatrix} 1 \\ 45 \\ 12 \end{pmatrix}$ is not well-defined.

Solution 1.2.

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solution 1.3. The product is AB and the result is

$$\begin{pmatrix} 32 & 17 & 24 & -1 \\ 1 & 1 & 0 & 1 \\ -18 & -10 & -9 & 1 \end{pmatrix}.$$

Solution 1.4.

- There exists such an integer for A:

  If A is a rotation by $2\pi p/n$ for a rational number p/n then $A^n = I$ because it corresponds to rotation by $2\pi p$.

- There does not exist such an integer for B: if B is a rotation by $2\pi q$ then B^n is a rotation by $2\pi qn$, which is never an integer multiple of 2π because q is irrational. (For example, if $2\pi qn = 2\pi N$ then $q = N/n$ so it would be rational). Therefore B^n is not the identity for any n.

- There exists such an integer for C, namely $C^2 = I$. This is because if you reflect twice then you get back to where you started. The mark is only given is there is a justification given. Don't give a mark if the justification picks a specific reflection and checks the matrix multiplication directly: the answer needs to work for all reflections. Students: I was happy with "reflecting twice gets you back to where you started", but if you want a more detailed answer, here you go. "Vectors along the line of reflection are fixed; vectors orthogonal to the line of reflection are multiplied by -1. Reflecting twice therefore sends all vectors back to where they started, so $C^2 = I$."

Solution 1.5. $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$GFG \neq 0$. For example, the vector $(0, 1)$ goes to $(1, 0)$ under G, then $(1, 0)$ under F, then $(0, -1)$ under G, so this composition is not zero.

$GFF \neq 0$. For example, the vector $(1,0)$ goes to itself under FF and then to $(0,-1$ under G.

$FGF = 0$ because this composition involves: projecting to the x-axis, then rotating to the y-axis, and then projecting back to the x-axis (which sends the y-axis to $(0,0)$).

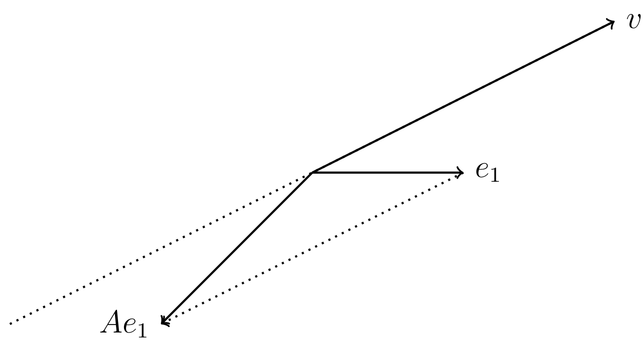
Solution 1.6. Let $A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$.

(a) Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$. If $Av = v$ then

$$-x + 4y = x, \qquad -x + 3y = y,$$

both of which reduce to $2y = x$. Therefore, for example, the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is fixed (so is any multiple of it).

(b) Here is a sketch of v, e_1, Ae_1:



(c) We see from our picture that e_1 is sheared backwards parallel to v.

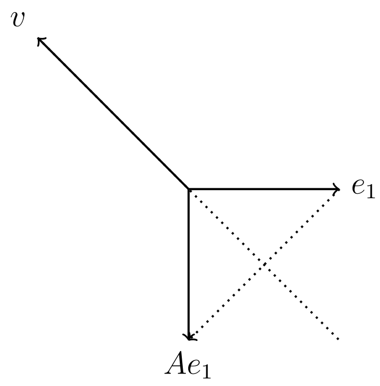
Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(a) Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$. If $Av = v$ then

$$-y = x, \qquad -x = y,$$

both of which reduce to $y = -x$. Therefore, for example, the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is fixed (so is any multiple of it).

(b) Here is a sketch of v, e_1, Ae_1:



(c) The picture clearly shows a reflection in the line v.

Solution 1.7. To find the axis, we need to solve $v = Mv$, i.e.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (x-z)/2 - y/\sqrt{2} \\ (x+z)/\sqrt{2} \\ (z-x)/2 - y/\sqrt{2} \end{pmatrix}.$$

This tells us that $y = (x+z)/\sqrt{2}$, so

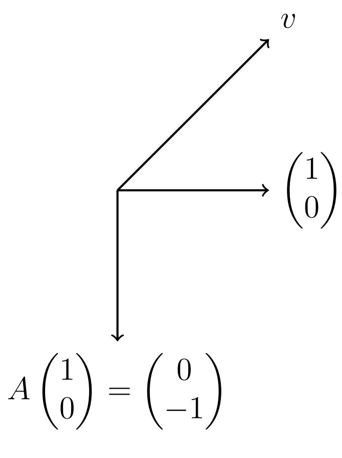
$$z = (z-x)/2 - (x+z)/2 = -x,$$

so the general solution is $(x, 0, -x)$. This means that the axis points in the $(1, 0, -1)$-direction.

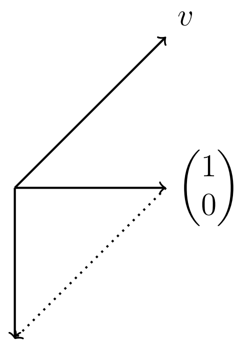
To calculate the angle, take the vector $w = (0, 1, 0)$ (orthogonal to the axis) and calculate the dot product $Mw \cdot w = (-1/\sqrt{2}, 0, -1/\sqrt{2}) \cdots (0, 1, 0) = 0$. This shows that the angle of rotation is 90 degrees (whether it's clockwise or anticlockwise depends on which way you orient your axis; they don't need to make a comment about this to get the marks).

Solution 1.8. (a) $Av = v$ means $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so $y = x$ and $2y - x = y$. In fact, $y = x$ implies $2y - x = y$, so $y = x$ is enough to have a solution. For example $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

(b) The vectors are:



The transformation A is a shear which fixes v and shunts $\begin{pmatrix}1\\0\end{pmatrix}$ downwards parallel to v.



$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

Solution 1.9. (a) False: e.g. $A = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ satisfies $A^2 = 0$.

(b) True: If $AB = I$ and $Bv = 0$ then $v = Iv = ABv = A0 = 0$.

Solution 1.10. (a) We have $dP/dx = 3a_3x^2 + 2a_2x + a_1$, whose vector of coefficients is $\begin{pmatrix}a_1\\2a_2\\3a_3\\0\end{pmatrix}$. Be careful: it's easy to get mixed up here. The first entry of the vector is the constant coefficient in dP/dx, i.e. a_1; the last entry is the coefficient of x^3, i.e. 0.

(b) We need the matrix ∂ such that

$$\partial\begin{pmatrix}a_0\\a_1\\a_2\\a_2\end{pmatrix} = \begin{pmatrix}a_1\\2a_2\\3a_3\\0\end{pmatrix},$$

so $\partial = \begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\\0 & 0 & 0 & 0\end{pmatrix}$.

(c) We have $\partial^2 v_p = \partial\begin{pmatrix}a_1\\2a_2\\3a_3\\0\end{pmatrix} = \begin{pmatrix}2a_2\\6a_3\\0\\0\end{pmatrix}$, which is the vector of coefficients of $6a_3x+2a_2$.

Note that this is equal to d^2P/dx^2. What we're seeing here is that differentiation is a linear operator, so can be represented as a matrix acting on vectors; for polynomials these vectors are only finite length, but for power series the vectors would be infinitely long...)

1.2 Solutions for enthusiasts

Solution 1.11.

$$\begin{aligned}
H_\phi^2 &= \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + (-\cos\phi)^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

using $\cos^2\phi + \sin^2\phi = 1$.

H_ϕ represents a reflection in the line parallel to $v_\phi = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}$. To see this, you first solve $H_\phi v = v$ to find fixed vectors (getting this precise v_ϕ involves some trig). Now the vector $w_\phi = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix}$ is (a) orthogonal to v_ϕ (because $v_\phi \cdot w_\phi = 0$) and (b) satisfies $H_\phi w_\phi = -w_\phi$. Therefore H_ϕ reflects vectors orthogonal to v_ϕ in the line v_ϕ.

Solution 1.12. $\exp\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\exp\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix}$.

$\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

$\exp\begin{pmatrix} 0 & ti \\ ti & 0 \end{pmatrix} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}$.

Solution 1.13. We have

$$\begin{aligned}
\frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} &= \frac{1}{\sqrt{\frac{(1+uv)^2 - (u+v)^2}{(1+uv)^2}}} \\
&= \frac{1+uv}{\sqrt{(1+uv)^2 - (u+v)^2}} \\
&= \frac{1+uv}{\sqrt{1 + u^2v^2 + 2uv - u^2 - v^2 - 2uv}} \\
&= \frac{1+uv}{\sqrt{1 + u^2v^2 - u^2 - v^2}} \\
&= \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}}.
\end{aligned}$$

Now

$$\begin{aligned}
\Lambda(u)\Lambda(v) &= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1+uv & -(u+v) \\ -(u+v) & 1+uv \end{pmatrix} \\
\Lambda\left(\frac{u+v}{1+uv}\right) &= \frac{1}{\sqrt{1-\left(\frac{u+v}{1+uv}\right)^2}} \begin{pmatrix} 1 & \frac{-(u+v)}{1+uv} \\ \frac{-(u+v)}{1+uv} & 1 \end{pmatrix} \\
&= \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1 & \frac{-(u+v)}{1+uv} \\ \frac{-(u+v)}{1+uv} & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1+uv & -(u+v) \\ -(u+v) & 1+uv \end{pmatrix}.
\end{aligned}$$

This shows $\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right)$.

This matrix is used in special relativity to transform from one reference frame to another which is moving with relative velocity v (working in units where the speed of light is 1, and with the simplifying assumption that space is 1-dimensional!). Naively, you would expect that if you increase the relative velocity by v and then by u, you would end up increasing it overall by $u+v$, but this formula shows that velocity addition is more subtle.

2 Week 2

2.1 Solutions

Solution 2.1. (a) C, D, F are in echelon form.

(b) D, F are in reduced echelon form.

(c) D has one free index (there is only one leading entry in column 1, so 2 is a free index). F has two free indices (2 and 3 are leading indices, so 1 and 4 are free).

Solution 2.2. The system was:

$$\begin{aligned} x - y + z &= -1 \\ 2x + z &= 5 \\ 3x + y - z &= 5 \end{aligned}$$

(a) To write this as a matrix equation $Av = b$, we need

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}, \qquad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad b = \begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix}.$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -1 & 5 \end{array}\right).$$

(b) Subtract twice row 1 from row 2 and three times row 1 from row 3 ($R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 3R_1$).

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 7 \\ 0 & 4 & -4 & 8 \end{array}\right)$$

Now subtract twice row 2 from row 3 $R_3 \mapsto R_3 - 2R_2$:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & -2 & -6 \end{array}\right)$$

Now we're in echelon form. To get to reduced echelon form, let's divide row 2 by 2 and row 3 by -2 ($R_2 \mapsto R_2/2, R_3 \mapsto R_3/2$):

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Now we add half row 3 to row 2 ($R_2 \mapsto R_2 + R_3/2$):

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

add row 2 to row 1 and subtract row 3 from row 1 ($R_1 \mapsto R_1 + R_2 - R_3$):

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The solution is now $x = 1, y = 5, z = 3$.

Solution 2.3. The answer is $c = -2$. To see this, write it as an augmented matrix:

$$\begin{pmatrix} 1 & 1 & -c & | & 1 \\ 0 & 1 & 1 & | & c \\ 1 & -1 & 0 & | & 2 \end{pmatrix}$$

Subtract row 1 from row 3 to get

$$\begin{pmatrix} 1 & 1 & -c & | & 1 \\ 0 & 1 & 1 & | & c \\ 0 & -2 & c & | & 1 \end{pmatrix}$$

Add twice row 2 to row 3 to get

$$\begin{pmatrix} 1 & 1 & -c & | & 1 \\ 0 & 1 & 1 & | & c \\ 0 & 0 & c+2 & | & 1+2c \end{pmatrix}$$

This is now in echelon form. The equation can be solved if and only if, for each row of zeros to the left of the bar, the term on the right of the bar is zero. The bottom row left of the bar is zero if and only if $c = -2$. So there is a solution when $c \neq -2$. When $c = -2$, the right hand side of the bar is $1 + 2c = -3 \neq 0$, so there is no solution.

Solution 2.4. Subtract twice row 1 from row 3 and three times row 1 from row 2:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & -6 & 1 \\ 0 & -2 & -3 & 1 \end{pmatrix}$$

Subtract row 2 from row 3:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & -6 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

There are three leading entries: $1, -2, 3$. So there are three leading indices ($1, 2, 3$, being the indices of the columns containing the leading entries).

Solution 2.5. The axis $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solves $Au = u$, i.e.

$$\begin{aligned} z &= x \\ -x &= y \\ -y &= x \end{aligned}$$

Therefore the axis points in the $\begin{pmatrix} x \\ -x \\ x \end{pmatrix}$-direction.

The vector $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is orthogonal to this axis, so we can compute the angle θ of rotation using

$$|v||Av|\cos\theta = v \cdot Av.$$

We have

$$Av = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix},$$

so

$$v \cdot Av = -1.$$

Finally, $|v| = |Av| = \sqrt{2}$, so $\cos\theta = 1/2$ and $\theta = 2\pi/3$ radians (120 degrees).

Solution 2.6. If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is orthogonal then $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$$a^2 + b^2 = 1, \quad bc = 0, \quad c^2 = 1.$$

The third equation implies $c = \pm 1$, so $bc = 0$ implies $b = 0$. Therefore $a^2 = 1$ and $a = \pm 1$. There are therefore four possibilities: $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

If $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ is orthogonal then similar reasoning shows $a = 0$ (and $b = \pm 1$).

Solution 2.7. (a) True: $(M + M^T)^T = M^T + (M^T)^T = M^T + M = M + M^T$.

(b) True: $(M - M^T)^T = M^T - (M^T)^T = M^T - M = -(M - M^T)$.

(c) True: $M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$.

2.2 Solutions for enthusiasts

Solution 2.8.

$$\begin{aligned} (A(BC))_{ij} &= \sum_k A_{ik}(BC)_{kj} \\ &= \sum_k \sum_\ell A_{ik}B_{k\ell}C_{\ell j} \\ ((AB)C)_{ij} &= \sum_k (AB)_{ik}C_{kj} \\ &= \sum_k \sum_\ell A_{i\ell}B_{\ell k}C_{kj} \end{aligned}$$

These agree when you rename $k \leftrightarrow \ell$.

Solution 2.9.

$$\begin{aligned}(A^T A)_{ij} &= \sum_k A^T_{ik} A_{kj} \\ &= \sum_k A_{ki} A_{kj}.\end{aligned}$$

The ith column of A is the vector v_i with kth component A_{ki} (remember the first index tells you what row you're on). Therefore $(A^T A)_{ij}$ is $v_i \cdot v_j$. Since $A^T A = I$, this implies the result.

Solution 2.10. (a) (i) $\exp(A)^T = \exp(A^T)$ follows by applying $(A^n)^T = (A^T)^n$ to each term in the power series.

(ii) We have

$$\begin{aligned}\frac{d}{dt}(\exp(tA)) &= \sum_{n=1}^{\infty} \frac{nt^{n-1}}{n!} A^n \\ &= A \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^{n-1} \\ &= A \exp(tA),\end{aligned}$$

where we have relabelled $m = n - 1$ in the sum.

(iii) A commutes with all the terms in the power series, hence with the power series itself.

(b) Differentiating $\exp(tA)^T \exp(tA) = I$ with respect to t gives:

$$A^T \exp(tA)^T \exp(tA) + \exp(tA)^T A \exp(tA) = 0.$$

Since $\exp(tA)^T \exp(tA) = I$, this simplifies to $A^T + A = 0$ (because $A \exp(tA) = \exp(tA) A$, so you can rewrite the last term as $\exp(tA)^T \exp(tA) A = IA = A$).

3 Week 3

3.1 Solutions

Solution 3.1. The inverse of A is $\begin{pmatrix} -13 & 2 \\ 7 & -1 \end{pmatrix}$ (if you're confused about signs, you probably forgot to divide by $ad - bc = -1$).

For $B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$, let's write the 4-by-8 augmented matrix:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right).$$

$R_2 \mapsto R_2 - R_1, R_3 \mapsto R_3 - 2R_1$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right).$$

$R_4 \mapsto R_4 - R_2$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right).$$

$R_3 \mapsto R_3 + R_4$, $R_2 \mapsto R_2 + 3R_4$, $R_1 \mapsto R_1 - R_4$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right).$$

$R_2 \mapsto R_2 - R_3$, $R_1 \mapsto R_1 - R_3$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right).$$

Now the left-hand block is in reduced echelon form, so the right-hand block is B^{-1}.

Solution 3.2. We need to put $\begin{pmatrix} 0 & -2 & 0 & 1 \\ 0 & -2 & -3 & 3 \\ 1 & 4 & 0 & -1 \\ 1 & 2 & -3 & 0 \end{pmatrix}$ into echelon form using only type I row operations (or, if we allow ourselves row-switches, we must keep track of whether we do an odd or even number).

We start with $R_1 \leftrightarrow R_3$ (ONE SWITCH!):

$$\begin{pmatrix} 1 & 4 & 0 & -1 \\ 0 & -2 & -3 & 3 \\ 0 & -2 & 0 & 1 \\ 1 & 2 & -3 & 0 \end{pmatrix}$$

Then $R_4 \mapsto R_4 - R_1$:

$$\begin{pmatrix} 1 & 4 & 0 & -1 \\ 0 & -2 & -3 & 3 \\ 0 & -2 & 0 & 1 \\ 0 & -2 & -3 & 1 \end{pmatrix}$$

Then $R_3 \mapsto R_3 - R_2$ and $R_4 \mapsto R_4 - R_2$:

$$\begin{pmatrix} 1 & 4 & 0 & -1 \\ 0 & -2 & -3 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

This is in echelon form. The determinant is therefore the product of the diagonal entries TIMES -1 because we did one row-switch. Thus the determinant is -12.

Solution 3.3. Let's write the 4-by-8 augmented matrix:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \mapsto R_2 + R_1$, $R_4 \mapsto R_4 + R_1$:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_3 \mapsto R_3 - R_2$, $R_4 \mapsto R_4 - R_2$:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$R_3 \leftrightarrow R_4$:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right)$$

We see that the determinant is 9: this is (minus) the product of the diagonal entries as we're in echelon form and have used one row-switch.

Continuing, $R_1 \mapsto R_1 + R_4$, $R_2 \mapsto R_2 + 2R_4$, $R_3 \mapsto R_3 - R_4$:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 0 & 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right)$$

$R_2 \mapsto R_2 + R_3$, $R_1 \mapsto R_1 + R_3$:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right)$$

$R_1 \mapsto R_1 - \frac{1}{3}R_2$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2/3 & -1/3 & 2/3 \\ 0 & 3 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right)$$

Finally, $R_2 \mapsto \frac{1}{3}R_2$, $R_3 \mapsto \frac{1}{3}R_3$, $R_4 \mapsto -R_4$ gives:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2/3 & -1/3 & 2/3 \\ 0 & 1 & 0 & 0 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & 0 & -1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \end{array}\right)$$

Solution 3.4. Any correct counterexample will do, but here are some possibilities below:

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfy $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) The common intersection of the four planes $z = 0$, $y = 0$, $y + z = 0$ and $y - z = 0$ is the x-axis.

(c) This is the same question, thinly disguised. The equations $z = 0$, $y = 0$, $y + z = 0$ and $y - z$ have $(x, 0, 0)$ as a solution for any x.

Solution 3.5. Let V be a linear subspace of $\mathbb{R}^n$ and $v \in V$ be a vector. We will show that $V \subset v + V$ and $v + V \subset V$, which will imply that $V = v + V$.

Proof that $V \subset v + V$: if $w \in V$ then $w - v \in V$ (as V is closed under addition) so $w = v + (w - v) \in v + V$.

Proof that $v + V \subset V$: if $w \in V$ then $v + w \in V$ because V is closed under addition).

Solution 3.6. The equations have solution $(x, 1 - x, 2 - x)$, which is a line (i.e. 1-dimensional affine subspace of $\mathbb{R}^3$). It doesn't pass through the origin, so it is not a linear subspace, therefore the solution is (d).

3.2 Solutions for enthusiasts

Solution 3.7. 1. The Jacobian matrix is

$$\begin{pmatrix} \frac{\partial(r\cos\phi)}{\partial r} & \frac{\partial(r\cos\phi)}{\partial\phi} \\ \frac{\partial(r\sin\phi)}{\partial r} & \frac{\partial(r\sin\phi)}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix},$$

whose determinant is $r\cos^2\phi + r\sin^2\phi = r$.

2. The Jacobian matrix is

$$\begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix},$$

whose determinant is (using the inductive formula, expanding along the bottom row):

$$\cos\theta(r^2\cos\theta\sin\theta\cos^2\phi + r^2\cos\theta\sin\theta\sin^2\phi) - (-r\sin\theta)(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi),$$

which simplifies to $r^2\sin\theta$.

Solution 3.8. 1. We have $1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$, so $\det(A) = \pm 1$.

2. We have

$$\begin{aligned} \det(A - I) &= \det(A(I - A^T)) \\ &= \det(A)\det(I - A^T) \\ &= \det(A)\det(I - A) \\ &= \det(A)\det(-I)\det(A - I) \\ &= (-1)^n \det(A - I), \end{aligned}$$

so if n is odd then $\det(A - I) = -\det(A - I)$, which means $\det(A - I) = 0$.

3. Since $\det(A - I) = 0$, $A - I$ is not invertible, so $(A - I)v = 0$ for some nonzero vector v. This means $Av = v$.

4 Week 4

Solution 4.1. The matrix is

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}.$$

One nice-looking column is column 3, with only one nonzero entry. Expanding, we get

$$\det(M) = 3 \times \det \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

Expanding this 3-by-3 matrix around the last row gives:

$$\det(M) = 3 \times \left(1 \times \det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} + 4 \times \det \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \right)$$

Evaluating the two 2-by-2 determinants gives:

$$\det(M) = 3 \times (1 + 4 \times 3) = 39.$$

Solution 4.2. The characteristic polynomial is

$$\det \begin{pmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 2 = \lambda^2 - 2\lambda + 3.$$

This has roots $\lambda_\pm = \frac{2 \pm \sqrt{4 - 4\times 3}}{2} = 1 \pm i\sqrt{2}$, which are then the eigenvalues. The eigenvectors are the solutions $v_\pm = \begin{pmatrix} x \\ y \end{pmatrix}$ of $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} v_\pm = \lambda_\pm v_\pm$, i.e.

$$x + 2y = (1 \pm i\sqrt{2})x, \qquad -x + y = (1 \pm i\sqrt{2})y,$$

so $y = \pm i\sqrt{2}x$. So the eigenvectors are $v_\pm = \begin{pmatrix} x \\ \pm i\sqrt{2}x \end{pmatrix}$.

Solution 4.3. The characteristic polynomial is

$$\det \begin{pmatrix} 4-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda) \det \begin{pmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 - 5\lambda + 6)$$

which has roots $1, 2, 3$.

Solution 4.4. To calculate the characteristic polynomial, we need to find the determinant of

$$M - \lambda I = \begin{pmatrix} 2-\lambda & -1 & c \\ -1 & 2-\lambda & -1 \\ c & -1 & 2-\lambda \end{pmatrix}.$$

We can, for example, expand across the top row using the cofactor expansion, which yields

$$-\lambda^3 + 6\lambda^2 + (c^2 - 10)\lambda + 4 + 2c - 2c^2.$$

0 is an eigenvalue if and only if $\det(M - 0I) = 0$, i.e. $4+2c-2c^2 = 0$. This is a quadratic equation with solutions $c = -1, 2$.

We find the eigenvectors by solving $Mv = 0$ with $c = -1$ and $c = 2$ respectively. This gives:

$$2x - y - z = 2y - x - z = 2z - x - y = 0 \Rightarrow x = y = z$$

(so $(1, 1, 1)$ or any nonzero multiple would be a valid answer) and

$$2x - y + 2z = 2y - x - z = 2x - y + 2z = 0 \Rightarrow y = 0, x = -z$$

(so $(1, 0, -1)$ or any nonzero multiple would be a valid answer).

Note from when this was an assessed question in 2021: Some people made errors early on which gave them the wrong values of c. As a result, they found that there were no nonzero eigenvectors. This should have been a sign that you had made an error: λ is an eigenvalue if and only if $Mv = \lambda v$ has a nonzero solution v. So if you're getting $v = 0$ as the only possibility, this is a sign you have picked the wrong eigenvalue (or, in this case, the wrong value of c). For this reason, the markers did not "carry errors" when marking this question.

Solution 4.5. (a) True. If A is lower-triangular then $A - \lambda I$ is also lower triangular, and hence its determinant is the product of its diagonal entries (this is the only way of picking n nonzero entries so that no two are in the same row and no two are in the same column). Therefore the characteristic polynomial of A is $(A_{11}-\lambda)\cdots(A_{nn}-\lambda)$, which has the diagonal entries of A as its roots.

(b) False. For example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda = 1$, $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mu = 2$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We get $A(v + w) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which is not a rescaling of $v + w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) True. The eigenvector equation is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which means $x + y + z = x$, $y + z = y$ so $z = y = 0$ and the only eigenvectors are multiples of $(1, 0, 0)$. Therefore the eigenspace is the x-axis, which is 1-dimensional.

Solution 4.6. (a) False. For example, the zero matrix (of any size) is in reduced echelon form but isn't invertible.

(b) True. Since $X^2 = -I$, we have $(-X)X = I$, so $X^{-1} = -X$.

(c) True. The characteristic polynomial has at least one (complex) root[1] λ, so $M - \lambda I$ fails to be invertible and hence annihilates at least one nonzero (complex) vector v, which is then a λ-eigenvector.

(d) False. For example, $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ sum to the identity. We have $\det(X) = \det(Y) = 0$ but $\det(X + Y) = 1$.

(e) True. Suppose $Xv = 0$. If X is invertible then $v = X^{-1}0 = 0$, so the only way to **1 marks**

[1] Any nonconstant polynomial has a root: this result is called the *fundamental theorem of algebra*.

have zero as an eigenvalue (i.e. to have a nonzero eigenvector with eigenvalue zero) is for X to be non-invertible.

Some people gave an alternative argument (also fine): the determinant of X is the product of its eigenvalues (taken with multiplicity). If one of these eigenvalues vanishes then $\det(X) = 0$, so X is not invertible.

Solution 4.7. You can check that (a) is an eigenvector with eigenvalue 3.

Solution 4.8. (a) Expand around the last row. We get

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This 4-by-4 matrix is upper triangular, so its determinant is the product of its diagonal entries, i.e. 1.

(b) This has repeated rows, so the determinant is zero.

(c) This is block-diagonal, so the determinant is the product

$$\det \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = (-1)(-1) = 1.$$

4.1 Solutions for enthusiasts

Solution 4.9. (a)

$$\frac{d^2P}{dx^2} - x\frac{dP}{dx} = 6a_3x + 2a_2 - 3a_3x^3 - 2a_2x^2 - a_1x,$$

which has $\begin{pmatrix} 2a_2 \\ 6a_3 - a_1 \\ -2a_2 \\ -3a_3 \end{pmatrix}$ as its vector of coefficients. Therefore the matrix corresponding to $P \mapsto \frac{d^2P}{dx^2} - x\frac{dP}{dx}$ is

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

(b) The characteristic polynomial is

$$\det \begin{pmatrix} -\lambda & 0 & 2 & 0 \\ 0 & -1-\lambda & 0 & 6 \\ 0 & 0 & -2-\lambda & 0 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix} = \lambda(\lambda+1)(\lambda+2)(\lambda+3),$$

so the eigenvalues are $0, -1, -2, -3$.

(c) The eigenvectors are:

$$\lambda = 0 : \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda = -1 : \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda = -2 : \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \lambda = -3 : \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

(d) The corresponding polynomials are

$$P_0 = 1, \qquad P_{-1} = x, \qquad P_{-2} = x^2 - 1, \qquad P_{-3} = x^3 - 3x.$$

(e) We have

$$\begin{aligned} \frac{d^2P_0}{dx^2} - x\frac{dP_0}{dx} &= 0 \\ \frac{d^2P_{-1}}{dx^2} - x\frac{dP_{-1}}{dx} &= -x = -P_{-1} \\ \frac{d^2P_{-2}}{dx^2} - x\frac{dP_{-2}}{dx} &= 2 - 2x^2 = -2P_{-2} \\ \frac{d^2P_{-3}}{dx^2} - x\frac{dP_{-3}}{dx} &= 6x - 3x^3 + 3x = -3P_{-3} \end{aligned}$$

as required.

This question was not completely random. These polynomials are called the *Hermite polynomials* and play an important role in the theory of spherical harmonics (this differential equation comes about when you separate variables in the Laplace equation in spherical coordinates). Through Schrödinger's equation and quantum mechanics, the eigenvalues are related to the energy levels in hydrogen atoms, and the eigenvectors are related to the probability amplitudes of electrons in those energy levels.

Solution 4.10. Since v is a λ-eigenvector, we have $w^TAv = \lambda w^Tv$. Since w is a μ-eigenvector, we have $w^TAv = (Aw)^Tv = \mu w^Tv$. Therefore

$$\lambda w^Tv = \mu w^Tv,$$

or $(\lambda - \mu)w^Tv = 0$ and either $\lambda = \mu$ or $v \cdot w = w^Tv = 0$, as required.

5 Week 5

5.1 Questions

Solution 5.1. The ellipsoid equation can be written as $v^TAv = 1$ for

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The eigenvalues of A are the solutions to $\det(A - tI) = 0$, i.e. $2, 2 \pm \sqrt{2}$. The eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix}$: these are the principal directions. The principal radii are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2\pm\sqrt{2}}}$.

Solution 5.2. The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are the solutions of $(a-t)^2 - b^2 = 0$, i.e. $t^2 - 2at + a^2 - b^2 = 0$. These solutions are $\frac{2a\pm\sqrt{4a^2-4a^2+4b^2}}{2} = a \pm b$. The corresponding eigenvectors are the solutions of

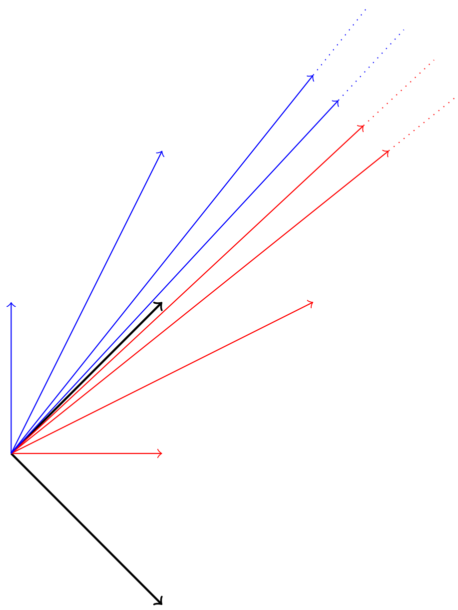
$$\begin{aligned} ax + by &= (a \pm b)x \\ bx + ay &= (a \pm b)y, \end{aligned}$$

which means $y = \pm x$. Therefore the eigenvectors are multiples of $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$.

We have $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

$$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} M^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(a+b)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(a-b)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so $M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gets closer and closer to the line $y = x$ as $n \to \infty$ (because $(a+b)^n \to \infty$ as $a + b > 1$ and $(a-b)^n \to 0$ as $0 < a - b < 1$; see the red vectors in the figure below). Similarly $M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gets closer to this line (blue vectors in the figure below).



Solution 5.3. The equation $Av = 0$ is

$$\begin{aligned} x + 7z &= 0 \\ 4x + 2y + z &= 0 \\ 3x + 2y - 6z &= 0 \end{aligned}$$

which has solution $x = -7z$, $y = 27z/2$, so the kernel consists of vectors of the form $\begin{pmatrix} -7z \\ 27z/2 \\ z \end{pmatrix}$. The nullity is 1 in this instance. By the rank-nullity theorem, the rank is 2.

The equation $Bv = 0$ is

$$\begin{aligned} 2x + 3y + w &= 0 \\ x + 2z &= 0, \end{aligned}$$

which has solution $x = -2z$, $w = 4z - 3y$, so the kernel consists of vectors of the form $\begin{pmatrix} -2z \\ y \\ z \\ 4z - 3y \end{pmatrix}$. The nullity is 2 in this instance. By the rank-nullity theorem, the rank is 2.

The equation $Cv = 0$ is

$$x - z = 0,$$

which has solution $z = x$, so the kernel consists of vectors of the form $\begin{pmatrix} x \\ y \\ x \end{pmatrix}$. The nullity in this instance is 2. By the rank-nullity theorem, the rank is 1.

Solution 5.4 (Solution to Exercise 5.4)**.** Suppose that $Av = \lambda v$. Consider the expression $\bar{v}^T Av$, where $\bar{v}$ denotes complex conjugation. Then, because $A = \bar{A}^T$, we have

$$\bar{\lambda}\bar{v}^T v = (\overline{Av})^T v = \bar{v}^T Av = \lambda \bar{v}^T v.$$

Note that if $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then $\bar{v}^T v = \sum |x_1|^2 + \cdots + |x_n|^2 > 0$ if $v \neq 0$, so dividing through by $\bar{v}^T v$ we get $\bar{\lambda} = \lambda$ and deduce that λ is real.

If v and w are two eigenvectors for distinct eigenvalues λ, μ then

$$\begin{aligned} \lambda \bar{w}^T v &= \bar{w}^T (Av) \\ &= \overline{(Aw)}^T v \\ &= \mu \bar{w}^T v \end{aligned}$$

(since $\bar{\mu} = \mu$ by the first part) so, since $\lambda \neq \mu$, we must have $\bar{w}^T v = 0$.

Solution 5.5. The Pell numbers satisfy

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix}.$$

(Let's write M for the matrix in this expression). The eigenvalues of M are the solutions of $0 = \det(M - tI)$, i.e.

$$0 = -t(2 - t) - 1 = t^2 - 2t - 1,$$

which are $\frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$. The eigenvectors are $\begin{pmatrix} 1 \\ 1 \pm \sqrt{2} \end{pmatrix}$. This means that $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 1 + \sqrt{2}$ (by the same argument that we used for the Fibonacci numbers in lectures).

Solution 5.6. We have

$$\det\begin{pmatrix} -t & -c_0 \\ 1 & -c_1 - t \end{pmatrix} = t^2 + c_1 t + c_0$$

$$\det\begin{pmatrix} -t & 0 & -c_0 \\ 1 & -t & -c_1 \\ 0 & 1 & -c_2 - t \end{pmatrix} = -(t^3 + c_2 t^2 + c_1 t + c_0)$$

$$\det\begin{pmatrix} -t & 0 & 0 & -c_0 \\ 1 & -t & 0 & -c_1 \\ 0 & 1 & -t & -c_2 \\ 0 & 0 & 1 & -c_3 - t \end{pmatrix} = t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0.$$

More generally, we have

$$\det\begin{pmatrix} -t & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & -t & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} - t \end{pmatrix} = (-1)^n (t^n + c_{n-1} t^{n-1} + \cdots + c_0).$$

5.2 Questions for enthusiasts

Solution 5.7. To prove our guess by induction, assume it's true for $n-1$ and let's evaluate the determinant

$$D_n := \det\begin{pmatrix} -t & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & -t & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} - t \end{pmatrix}.$$

Expanding along the first column, we get

$$-tD_{n-1} - \det\begin{pmatrix} 0 & \cdots & \cdots & 0 & -c_0 \\ 1 & -t & & 0 & -c_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & -t & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} - t \end{pmatrix}$$

We evaluate this final determinant as follows: pick c_0 from the top row (everything else is zero); pick either 1 or $-t$ from the second row; if you picked 1 you can pick either 1 or $-t$ from the third row whereas if you picked $-t$ you have to pick $-t$ from the third row; continue in this manner and you either pick all the 1s or else you pick $-t$ until you reach the last row and you're forced to pick a zero. Therefore the only term we get is c_0 (with a sign). The sign is $(-1)^{n+1}$ as we can see because this permutation is $(123\cdots n)$. Therefore

$$D_n = -tD_{n-1} - (-1)^{n+1} c_0 = (-1)^n (t^n + c_{n-1} t^{n-1} + \cdots + c_0)$$

by inductive hypothesis (i.e. substituting in what we assumed was the formula for D_{n-1}). We already checked the base case D_2, so we're done.

Solution 5.8. We have

$$\det\begin{pmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{pmatrix} = \lambda_2 - \lambda_2.$$

Next:

$$\det\begin{pmatrix}1 & \lambda_1 & \lambda_1^2\\ 1 & \lambda_2 & \lambda_2^2\\ 1 & \lambda_3 & \lambda_3^2\end{pmatrix} = \lambda_2\lambda_3^2 - \lambda_2^2\lambda_3 - \lambda_1(\lambda_3^2 - \lambda_2^2) + \lambda_1^2(\lambda_3 - \lambda_2)$$

i.e. $\lambda_2\lambda_3(\lambda_3 - \lambda_2) - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 + \lambda_2) + \lambda_1^2(\lambda_3 - \lambda_2)$, so we can pull out a factor of $\lambda_3 - \lambda_2$, leaving

$$\lambda_2\lambda_3 - \lambda_1\lambda_3 - \lambda_1\lambda_2 + \lambda_1^2 = (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$$

and again we get $\det(V) = Q$ as required.

If $\lambda_k = \lambda_\ell$ then the Vandermonde determinant vanishes (two rows coincide) and the expression Q vanishes (because $\lambda_k - \lambda_\ell$ is a factor).

Solution 5.9. We saw that the characteristic polynomial of C is $\pm(t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$, so the eigenvalues of C are the roots $\lambda_1, \ldots, \lambda_n$ of this polynomial. If we form the corresponding Vandermonde matrix whose ith row is given by powers of λ_i then we see that

$$\begin{aligned}\begin{pmatrix}1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1}\end{pmatrix} C &= \begin{pmatrix}\lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} & -c_0 - c_1\lambda_i - \cdots - c_{n-1}\lambda_i^{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} & \lambda_i^n\end{pmatrix}\\ &= \lambda\begin{pmatrix}1 & \lambda & \cdots & \lambda^{n-1}\end{pmatrix}.\end{aligned}$$

Therefore the ith row of VC is λ_i times the ith row of V. Let R_i be the ith row of V and C_i be the ith columns of V^{-1} (so $R_iC_j = \delta_{ij}$). Then VC has rows $\lambda_i R_i$, so the ij entry of VCV^{-1} is $\lambda_i R_i C_j = \lambda_i \delta_{ij}$, and we see that VCV^{-1} is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ down the diagonal.

We know that $\mathrm{Tr}((VCV^{-1})^m) = \mathrm{Tr}(VC^mV^{-1}) = \mathrm{Tr}(V^{-1}VC^m) = \mathrm{Tr}(C^m)$ because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ from sheet 1. Therefore $\mathrm{Tr}(C^m) = \mathrm{Tr}(diag(\lambda_1^m, \ldots, \lambda_n^m)) = \sum \lambda_i^m$.

Solution 5.10. By the previous question, we know that the sum of squares of the roots of this polynomial is $\mathrm{Tr}(C^2)$. The only nonzero diagonal entries of C^2 are the final two, which are $-c_{n-2}$ and $c_{n-1}^2 - c_{n-2}$, so we get $c_{n-1}^2 - 2c_{n-2}$ as required. For the sum of cubes, we compute $\mathrm{Tr}(C^3)$ and we get $-c_{n-1}^3 + 3c_{n-2}c_{n-1} - 3c_{n-3}$.

Solution 5.11. In the case $m = 2, n = 1$ the Sylvester matrix is

$$\begin{pmatrix}p_2 & p_1 & p_0\\ q_1 & q_0 & 0\\ 0 & q_1 & q_0\end{pmatrix},$$

whose determinant is $p_2q_0^2 - p_1q_1q_0 + p_0q_1^2$.

If there exist a, b such that $p = (a + bz)q$ then

$$p_2 = bq_1, \quad p_1 = aq_1 + bq_0, \quad p_0 = aq_0,$$

so the determinant is

$$bq_1q_0^2 - (aq_1 + bq_0)q_1q_0 + aq_0q_1^2 = 0.$$

Conversely, suppose that $p_2q_0^2 - p_1q_1q_0 + p_0q_1^2 = 0$. Note that $q_1 \neq 0$ or else q has degree less than 1.

- If $q_0 = 0$ then $p_0q_1^2 = 0$, so $p_0 = 0$ and $z|p(z)$. Since $q_0 = 0$, $q(z) = q_1z$, so $z|q(z)$ and we have found a common factor.
- if $q_0 \neq 0$ then consider the polynomial $(p_2/q_1)z + (p_0/q_0)$. We have

  $$(p_2/q_1)z + (p_0/q_0)(q_1z + q_0) = p_2z^2 + \left(\frac{p_2q_0}{q_1} + \frac{p_0q_1}{q_0}\right) + p_0$$

  and the term in brackets is equal to p_1 because $p_2q_0^2 + p_0q_1^2 = p_1q_0q_1$. Therefore q is a common factor of p and q.