Lecture XI: The non-Kähler world

Jonathan Evans

2nd December 2010

We've spent most of the course so far discussing examples of symplectic manifolds coming from algebraic geometry and exploring their topology. This seems like a good idea: the topology of objects arising in algebraic geometry is usually highly nontrivial (e.g. Hopf fibration, exotic spheres coming from links of singularities) yet explicit. However in doing so we're cutting ourselves off from the vast majority of the symplectic world, as we'll see in this lecture. The aim is to introduce techniques to construct and detect non-Kähler symplectic manifolds, and thereby open our eyes to the startling diversity of the class of all symplectic manifolds.

What's so special about Kähler manifolds?

Recall that

Definition

A symplectic manifold (X, ω) *is called Kähler if it admits an* ω*-compatible integrable complex structure* J*. The associated metric* $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ *is called the Kähler metric.*

If we want to find a manifold which is symplectic but non-Kähler then we need criteria by why we can rule out the existence of a compatible complex structure, i.e. we need topological restrictions on Kähler manifolds. The most basic of these comes from the symmetries of the Hodge diamond...

Theorem (Hodge)

A Kähler manifold admits a decomposition of its cohomology with complex coefficients

$$H^k(X;\mathbb{C}) \cong \bigoplus_{i+j=k} H^{i,j}(X)$$

where $H^{i,j}(X)$ *is the* (i,j)*-th Dolbeaut cohomology group* $H^i(X,\Omega^j)$*. Moreover we have symmetries*

$$H^{i,j}(X) \cong \overline{H^{j,i}(X)}$$
$$H^{n-i,n-j}(X) \cong H^{i,j}(X)^*$$

coming from complex conjugation and Serre duality respectively.

Corollary

$\dim H^{1,0}(X) = \dim H^{0,1}(X)$ *so* $b_1 = \dim H^{1,0}(X) + \dim H^{0,1}(X)$ *is even for a Kähler manifold. Indeed* b_j *is even for any odd* j*.*

Kodaira-Thurston manifold

Here is an example of a symplectic 4-manifold with odd first Betti number. Let $K = T^3 \times \mathbb{R}/\mathbb{Z}$ where $j \in \mathbb{Z}$ acts by

$$(x, y, z, t) \mapsto (x, y + jx, z, t + j)$$

The symplectic form $dx \wedge dy + dz \wedge dt$ is invariant under the diffeomorphism $(x, y, z, t) \mapsto (x, y + jx, z, t + j)$ and hence descends to give a symplectic form on K. The space K has a bundle projection to the (z, t)-torus and the fibre is a symplectic T^2 (can you write it as a fibre bundle with Lagrangian torus fibres?). The universal cover is $\mathbb{R}^4$ with deck group

$$\Gamma = \{(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4\}, (j', k') \star (j, k) = (j + j', A_{j'}k + k')$$

where A_j is the matrix

$$A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix}$$

so $\pi_1(K) = \Gamma$. Notice that the commmutator subgroup is $[\Gamma, \Gamma] \cong \mathbb{Z}$, generated by $(0, 0, 1, 0)$. Therefore b_1, the rank of $H_1 = \Gamma/[\Gamma, \Gamma]$, is 3.

McDuff's example

Here's a simply-connected example of a non-Kähler symplectic manifold due to McDuff. We haven't really developed enough theory to explain this one properly, so you'll have to take some stuff on trust, namely.

Theorem (Gromov-Tischler)

It's possible to embed a symplectic k-manifold X symplectically in $\mathbb{CP}^{k+1}$.

So embed K symplectically in $\mathbb{CP}^5$ and blow it up to get a 10-manifold $\widetilde{\mathbb{CP}^5}$ (recall that blowing up a submanifold means replacing each fibre V of its normal bundle with the projective space $\mathbb{P}V$). We know that blowing up a point doesn't affect the fundamental group. In fact the same is true for blowing-up submanifolds, so we know that $\pi_1(\widetilde{\mathbb{CP}^5}) = \{1\}$. When we blow up, a Mayer-Vietoris calculation shows that $b_3 = H_3(\widetilde{\mathbb{CP}^5}) = H_1(K)$ which is odd. Therefore McDuff's example cannot be Kähler.

π_1(Kähler)?

Can we characterise the topology of a Kähler manifold? This is a hard and venerable question. Other than the $b_1 \equiv 0 \mod 2$ restriction coming from a naive inspection of the Hodge diamond, there are other classical restrictions on the topology (formality) coming from the hard Lefschetz theorem and one can cook up symplectic manifolds which violate these. Instead of discussing this (which would require quite a detour) there's another incarnation of this venerable question which is more concrete and whose symplectic analogue fails utterly.

Question

Which finitely-presented groups arise as fundamental groups of Kähler manifolds?

Answer

The answer is not known, but there are extremely subtle restrictions coming from Hodge theory and harmonic map theory. For example, if $G = \pi_1(H)$ *for a hyperbolic* $n > 2$*-manifold* $H = \mathbb{H}/G$ *(i.e. quotient of hyperbolic* n*-space* $(n > 2)$ *by cocompact discrete subgroup of isometries) then* $G \neq \pi_1(X)$ *for any Kähler* X*. To see this, suppose* X *is Kähler and* $\pi_1(X) = G$*. Then there is a "classifying map"* $X \to \mathbb{H}/G = K(G, 1)$ *for the fundamental group. Approximate this by a smooth map and then apply harmonic map flow to it. Negative curvature of the target space means the flow is well-behaved and you end up with a harmonic classifying map. Now a very clever differential geometry argument shows that the differential of a harmonic map from a Kähler manifold to a negatively curved manifold has rank at most 2 (see Carlson-Toledo).*

Gompf's argument

So which groups can occur as $\pi_1(X)$ where X is a symplectic manifold? More groups than for Kähler manifolds? Yes: even beyond the $b_1 \equiv 0 \mod 2$ examples. For example one can relatively easily construct symplectic manifolds with hyperbolic fundamental groups (the twistor spaces of hyperbolic manifolds) which are therefore non-Kähler. More shockingly

Theorem (Gompf)

For any $n \geq 2$, any finitely-presented group can occur as the fundamental group of a symplectic $2n$-manifold.

But Gompf's result really illustrates how big the class of symplectic manifolds is and indicates how hopeless any attempt at classification would be (bear in mind that there are logical obstructions to the algorithmic classification of finitely-presented groups: the symplectic manifolds is even bigger!).

Gompf's argument is based on a surgery procedure called fibre connect-sum (or just Gompf sum).

- Ordinary connect sum in 2-d: You take a point $p \in X$ and $q \in Y$, you find small radius ϵ Darboux charts B and C centred at p and q, you identify $B \setminus \{p\} \cong B_\epsilon \setminus \{0\}$ with $C \setminus \{q\} \cong B_\epsilon \setminus \{0\}$ via a symplectic involution ϕ of the annulus $B_\epsilon \setminus \{0\}$ which switches the boundary components. This allows you to form the union

$$X \setminus \{p\} \coprod Y \setminus \{q\} / \sim, \; x \sim \phi(x)$$

  which is symplectic because the gluing map on the overlap is symplectic. The symplectic form on the annulus is $rdr \wedge d\theta$ and ϕ can be taken to be

$$(r, \theta) \mapsto (\sqrt{\epsilon^2 - r^2}, -\theta)$$

- We cannot mimic this in higher dimensions: otherwise we could use ϕ to glue two symplectic discs boundary to boundary and get a symplectic structure on the sphere S^n, $n > 2$, which has $H^2(S^n) = 0$, so is not symplectic.

In the ordinary connect sum we used the fact that a point has real codimension 2 so its punctured normal bundle is an annulus. In higher dimensions we might hope to mimic the connect sum if we had a real codimension 2 symplectic submanifold Σ: then its punctured normal bundle (i.e. $\nu\Sigma \setminus \Sigma$) is fibred by annuli and we could use ϕ fibrewise. This works and is what we call *fibre sum* or *Gompf sum*. We need:

- A pair of symplectic manifolds X, Y of the same dimension $2n$,
- In each of X and Y a symplectic submanifold (A and B respectively) and a symplectomorphism $\psi : A \to B$,
- An isomorphism of symplectic normal bundles $\tilde{\psi} : \nu A \cong \nu B^*$ which covers the symplectomorphism ψ.

We took the dual νB^*: why do we want opposite normal bundles? This is just the usual thing with gluing: you want to think of A as being "at infinity" in X and B as being "at zero" in Y; now think about how the oriented unit circle looks like from 0 and from ∞ in $\mathbb{CP}^1$.

More explicitly, we pick a fibrewise compatible almost complex structure on νA and push it forward to νB along $\tilde{\psi}$. This lets us talk about radius of normal vectors (remember the normal fibre has a symplectic form because it's the symplectic orthogonal complement of a symplectic subspace). We also pick a 1-form τ on $\nu A \setminus A$ which evaluates to 1 on $\partial/\partial\theta$ and to 0 on radial vectors in each fibre (this is equivalent to picking a connection: we require τ to vanish also on horizontal vectors). Then $\eta = \frac{1}{2}d(r^2\tau)$ is a 2-form on $\nu A \setminus A$ which equals $rdr \wedge d\theta$ on each fibre. Now $\Omega_A = t\eta + \pi^*\omega_A$ (and $(\tilde{\psi}^{-1})^*\Omega_A = \Omega_B$) is a symplectic form on νA (respectively νB) for small t and the symplectic neighbourhood theorem tells us that $(\nu A, \Omega_A)$ is symplectomorphic to a neighbourhood of $A \subset X$ (same for B). Now we glue these neighbourhoods by using the fibrewise symplectomorphism ϕ, noting that ϕ reverses the orientation of the normal fibres.

Note that the choice of $\tilde{\psi}$ is not unique, even up to homotopy. Using different isomorphisms will give (even topologically) different *twisted* fibre sums: e.g. connect summing $T^2 \times S^2$ with $T^2 \times T^2$ along a T^2 can give T^4 if done in the obvious way, but can give the Kodaira-Thurston manifold K if a twisted gluing map is used. Also, the parameter ϵ (or t) determining the size of the neighbourhoods we chose determines the total volume of the fibre sum. Different ϵs give different volumes.

Example (Elliptic surfaces)

Consider a generic pencil of cubic curves in $\mathbb{CP}^2$*. Two generic cubics intersect in nine points so the base locus of the pencil consists of nine points. Write* $E(1)$ *for the blow-up of* $\mathbb{CP}^2$ *along the base locus. This admits a map to* $\mathbb{CP}^1$ *(with twelve singular fibres) whose smooth fibres are cubic curves (symplectic tori). The normal bundle to a fibre is trivial (because the fibre has zero intersection with nearby (homologous) fibres). Therefore we can fibre sum* $E(1)$ *(rational elliptic surface) with itself along a fibre* k *times and we get a symplectic manifold* $E(k)$*. It's not hard to compute the Euler characteristic of* $E(k)$ *is* $12k$ *and to see (van Kampen) that* $E(k)$ *is simply-connected. In fact,* $E(2)$ *is a symplectic K3 surface.*

Proof of Gompf's theorem

Given an arbitrary finitely presented group

$$G = \langle \alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_\ell \rangle$$

we will now construct a symplectic 4-manifold[1] (X, ω) with $\pi_1(X) \cong G$. To do this we start with a symplectic manifold with even bigger fundamental group:

$$(F \times T^2, \omega_F \oplus \omega_{T^2})$$

where F is a surface of genus k. Recall that

$$\pi_1(F) = \langle a_1, \ldots, a_k, b_1 \ldots, b_k | \prod_{i=1}^{k} [a_i, b_i] = 1 \rangle$$

Think of $\langle \alpha_1, \ldots, \alpha_k \rangle$ as the free subgroup generated by $a_1, \ldots, a_k$. Let $\{\gamma_i\}_{i=1}^{\ell}$ be immersed loops representing homotopy classes corresponding to the relations $\{\beta_i\}_{i=1}^{\ell}$ and let $\gamma_{\ell+i} = b_i$ for $i = 1, \ldots, k$. Pick an embedded homologically essential loop $\lambda \subset T^2$ and consider the Lagrangian tori

$$T_i = \gamma_i \times \lambda \subset F \times T^2$$

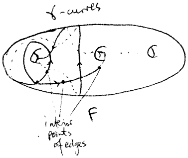
[1] To get 6-, 8-,... manifolds just multiply with $S^2 \times S^2 \times \cdots$.

We will perturb the symplectic form to make sure these are symplectically immersed and then perturb them to ensure they're symplectically embedded and mutually disjoint. The idea will be to fibre sum each of these with $E(1)$. In the resulting manifold the fundamental group will have changed by killing off the word γ_i. Performing one further surgery to a symplectic torus $\{z\} \times T^2$ allows us to kill off the generators of the fundamental group coming from the torus factor, leaving us with G.

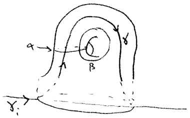
The fact that the surgery kills the relevant elements of the fundamental group will follow from van Kampen's theorem once we know that $E(1) \setminus T^2$ is simply-connected. To see this, observe (van Kampen) that its fundamental group is generated by a meridian for the T^2 (since $E(1)$ is simply-connected). But the 9 exceptional spheres are sections of the Lefschetz fibration $f : E(1) \to \mathbb{CP}^1$ and a meridian just projects to a circle encircling the point $f(T^2)$. Therefore we can take as our meridian the image of this circle under a section. But this is nullhomotopic in the complement $E(1) \setminus T^2$, using the other hemisphere of the section as a nullhomotopy.

We will now perturb the symplectic form. Suppose ρ be a closed 1-form on F which restricts to a volume form on each of the oriented loops γ_i (we will construct this in a moment) and let θ be a 1-form on T^2 which is a volume form on λ. The 2-form $\omega + \epsilon\rho \wedge \theta$ is symplectic form small ϵ (since the nondegeneracy condition is open) and makes T_i into a symplectically immersed submanifold. Identify T^2 with $\lambda \times S^1$ and note that $T_i \subset F \times \lambda$. We can resolve each T_i near its immersed points by perturbing it in the S^1-direction and ensuring it's embedded. Similarly we can disjoin the T_i from one another.

Now we will construct ρ. Note that we may have to modify the surface to achieve this. First perturb the curve γ so they all intersect transversely and their union is a graph in F. We will find a closed 1-form ρ' whose integral along each edge is positive. Once we have this we know that $\gamma_i^*\rho'$ is homologous to a volume form θ_i and $\theta_i - \gamma_i^*\rho' = df_i$ for some function f_i. We can assume each f_i vanishes at the vertices of the graph and this allows us to extend them to a function $f : F \to \mathbb{R}$. Finally we let $\rho = \rho' + df$. There may be topological obstructions to finding ρ' in the first place. For example if the graph contains a closed cycle bounding a polygon then the integral of $0 = d\rho'$ over the polygon equals the sum of the integrals of ρ' along the edges by Stokes' theorem, which should be positive by assumption. To construct ρ' we modify F. This construction is sketched in the following slide.

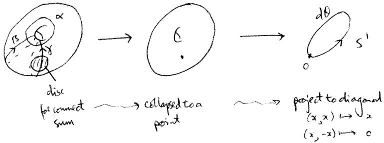


For each edge in the graph of γ curves, pick an interior point and connect sum with a torus



New surface F' and new graph (including α, β and $\gamma_i \# \gamma$)

$\rho' = 0$ away from torus handles.



CONSTRUCTING ρ' on each handle: pull back $d\theta$ along the collapse-project maps.

- ρ' is smooth (collapse map means that ρ' extends smoothly by 0 to complement of handles)
- $\int \rho' > 0$ along each edge (each edge enters a handle where indeed the integral is positive)

We are drawing close to the end of this short course in symplectic topology and we have so far proved two “hard theorems”: Luttinger unknottedness and Gromov’s nonsqueezing. In both cases we just assumed the results we needed from the theory of pseudoholomorphic curves. This unsatisfactory state of affairs must continue one lecture longer because there is another central topic in symplectic topology which I do not feel I could omit from an introductory course, namely the theory of Hamiltonian group actions and symplectic reduction. It will hopefully only take me one lecture to dispense with my guilt at not mentioning this subject, leaving us with two lectures in which I will outline the main theorems about pseudoholomorphic curves and how to use them.