Lecture V: Lagrangians II

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- Last time we talked about Lagrangian submanifolds $L \subset X$, about how they admit neighbourhoods symplectomorphic to neighbourhoods of the zero-section in T^*L and about the restrictions this places on their topology.
- I finished with the promise of proving Luttinger's unknottedness theorem, which asserts the existence of knotted tori in $\mathbb{C}^2$ which are not isotopic to any Lagrangian embedding and that is what we'll prove today.
- We'll then go on and fill in some of the gaps that we've left so far, including a discussion of Chern classes and adjunction for symplectic submanifolds, and remind ourselves about the topology of the Lagrangian Grassmannian which I did rather quickly at the end of Lecture II.

Theorem (Luttinger)

No nontrivial spin knot torus is isotopic to a Lagrangian embedding in $\mathbb{C}^2$.

A spin knot torus is what you get by taking a nontrivial knot inside a 3-dimensional half-space H in $\mathbb{C}^2$ and rotating it around the axis ∂H. It traces out a knotted torus in $\mathbb{C}^2$. Luttinger used a surgery procedure to prove this, which is the analogue of Dehn surgery for Lagrangian 2-tori. Therefore to begin we'll review Dehn surgery of 3-manifolds

Dehn surgery

- Given a knot $\iota : K \to M$ in a 3-manifold M, let νK denote its tubular neighbourhood and T denote the boundary $\partial \nu K$. $\nu K \cong K \times D^2$ and $T \cong K \times S^1 = T^2$.
- We can identify curves on T: the meridian is the curve $\gamma = \{\star\} \times S^1$, a longitude is a curve which projects down with degree 1 to K under the normal bundle projection.
- While there is only one possible choice of γ up to homotopy (given by the image of a unit normal circle under the exponential map for some metric) there are many choices of longitude. For example, given a longitude λ, you can modify it to another longitude which winds around in the meridian direction k times (i.e. represents the class $[\lambda] + k[\gamma]$. A choice of longitude is called a *framing*.

- An oriented knot in $\mathbb{R}^3$ has a canonical framing λ_S defined as follows. Let S be an orientable surface bounding K (a Seifert surface) - which exists because K is nullhomologous - and let λ be a small pushoff of K inside Σ.
- Another Seifert surface S' will give the same framing up to homotopy. To see this, note that by Mayer-Vietoris there is a natural isomorphism

$$0 \to H_1(T^2; \mathbb{Z}) \to H_1(\mathbb{R}^3 \setminus K; \mathbb{Z}) \oplus H_1(K; \mathbb{Z}) \to 0$$

  and the Seifert framing is the longitude $[\lambda] \in H_1(T^2; \mathbb{Z})$ corresponding to $(0, 1)$ (nullhomologous in the knot complement, projects with degree 1 onto K).
- For coprime integers p, q the (p, q)-Dehn surgery on K is the manifold

$$\mathbb{R}^3_{p,q}(K) = \left((\mathbb{R}^3 \setminus \nu K) \coprod \nu K\right) / \{x \equiv \phi(x) : x \in T\}$$

  where ϕ is the linear automorphism of T sending γ to $p\lambda_S + q\gamma$.

Luttinger surgery

The proof uses a surgery construction on Lagrangian tori similar to Dehn surgery on knots in 3-manifolds. This is due to Luttinger, but here we present it in a more explicit form given by Auroux, Donaldson and Katzarkov.

- Let $\iota : T^2 \to X$ be a Lagrangian embedding of a torus in (X, ω). Weinstein's neighbourhood theorem gives us an extension $\tilde{\iota} : U = D^*T^2 \to X$ where $D^*T^2 \cong D^2 \times T^2$ is a disc-subbundle of the cotangent bundle.
- Pick coordinates $(q_1, q_2) \in T^2$ and (p_1, p_2) on D^2 so that $\omega = \sum dq_i \wedge dp_i$.
- Consider a small ϵ such that $[-\epsilon, \epsilon]^2 \subset D^2$. Write $U_\epsilon = T^2 \times [-\epsilon, \epsilon]^2$.

- Let $\chi : [-\epsilon, \epsilon] \to [0,1]$ be a smooth step function equal to 0 for $t \leq -\epsilon/3$ and to 1 for $t \geq \epsilon/3$. Suppose moreover that

$$\int_{-\epsilon}^{\epsilon} t\chi'(t)dt = 0$$

- For $k \in \mathbb{Z}$ define a symplectomorphism

$$\phi_k : U_\epsilon \setminus U_{\epsilon/2} \circlearrowleft$$

by

$$\phi_k(q_1, q_2, p_1, p_2) = (x_1 + k\chi(y_1), x_2, y_1, y_2) \text{ if } y_2 \geq \epsilon/2$$
$$\phi_k = \mathrm{id} \text{ otherwise.}$$

Definition

Given a Lagrangian embedding $\iota : T^2 \to X$, a choice of coordinates (q_1, q_2) on T^2 and a Weinstein neighbourhood $U_\epsilon \to X$ of ι the k-framed Luttinger surgery on ι is the manifold

$$X_k(\iota) := (X \setminus U_{\epsilon/2}) \cup_{\phi_k} U_\epsilon = \left((X \setminus U_{\epsilon/2}) \coprod U_\epsilon \right) / \left\{ u \sim \phi_k(u), \; u \in U_\epsilon \setminus U_{\epsilon/2} \right\}$$

where ϕ_k is understood as a gluing map on the overlap. Since ϕ_k is a symplectomorphism, the symplectic forms on each part of the manifold agree on the overlap.

We never actually used the condition $\int_{-\epsilon}^{\epsilon} t\chi'(t)dt = 0$. This is used in Auroux-Donaldson-Katzarkov to prove that the construction is independent of choices we made and to investigate how the surgered manifold depends on the original torus.

- To do this surgery topologically we don't need the torus to be Lagrangian. All we need is a framing.
- Of course the resulting manifold won't necessarily be symplectic. If the surgered manifold is not symplectic then we know that the torus and the framing do not come from a Weinstein framing of a Lagrangian torus.
- We want to rule out certain knotted tori from having Lagrangian representatives. The idea will be to do topological Luttinger surgery and show that the resulting manifold cannot be symplectic, but we also have to pick a framing.
- It would help if we could characterise Weinstein framings topologically...

Now the knot K is a torus in $\mathbb{C}^2$, the normal bundle of the knot is $\nu K = T^2 \times D^2$ and the boundary $T = \partial \nu K$ is a 3-torus. Mayer-Vietoris gives an isomorphism

$$0 \to H_1(T;\mathbb{Z}) \to H_1(\mathbb{C}^2 \setminus K;\mathbb{Z}) \oplus H_1(K;\mathbb{Z}) \to 0$$

and there is a canonical meridian $\gamma \in H_1(T;\mathbb{Z})$. A framing is a choice of $\alpha, \beta \in H_1(T;\mathbb{Z})$ which project to generators a, b in $H_1(K;\mathbb{Z})$. Given a choice of such generators, there is a canonical *Seifert framing* which corresponds to $(0,(a,b))$ under this isomorphism.

Proposition

The Weinstein framing of a Lagrangian torus is a Seifert framing, that is the longitudes are nullhomologous in the knot complement.

To prove this we introduce a very large (but necessary) sledgehammer.

Theorem (Gromov)

Let X be a symplectic manifold containing a compact set K such that $X \setminus K$ is symplectomorphic to a standard ball complement in $\mathbb{C}^2$. Then X is symplectomorphic to (a blow-up of) $\mathbb{C}^2$.

This is hard and uses pseudoholomorphic curve theory. Blow-up is something we'll see in a later lecture: it's irrelevant here because blowing up $\mathbb{C}^2$ k times gives a manifold with signature k but Luttinger surgery gives a manifold with Euler characteristic zero (by additivity of Euler characteristic). The theorem implies

Corollary

If $L \subset \mathbb{C}^2$ is an embedded Lagrangian torus then $\mathbb{C}^2_k(L)$ is symplectomorphic to $\mathbb{C}^2$. In particular it's simply-connected.

This corollary will be used to prove both the proposition on framings and the unknottedness theorem.

Proof of Proposition.

Suppose α and β are a Weinstein framing. We can perform k-Luttinger surgery along α using this framing. Since the surgered manifold has no higher homology by Gromov's theorem, Mayer-Vietoris implies that

$$H_1(\partial\nu T^2;\mathbb{Z}) \xrightarrow{-\Phi\oplus\Psi} H_1(\mathbb{C}^2\setminus T^2;\mathbb{Z})\oplus H_1(T^2;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}^2$$

is an isomorphism. But relative to the basis $\{\alpha,\beta,\gamma\}$ of $H_1(\partial\nu T^2;\mathbb{Z})$ the matrix of $-\Phi\oplus\Psi$ is

$$\begin{pmatrix} -1 & 0 & k \\ 0 & -1 & 0 \\ A & B & 1 \end{pmatrix}$$

Being an isomorphism of $\mathbb{Z}$-modules we need $\det(-\Phi\oplus\Psi)=\pm1$ but we have

$$\det(-\Phi\oplus\Psi)=1-Ak$$

for all k, so $A=0$. Similarly $B=0$. This proves that the pushoffs of curves in T^2 under a Weinstein framing in $\mathbb{C}^2$ are precisely the nullhomologous ones in the knot complement. □

- Now suppose that we have a spin knot L, i.e. obtained from a knot K in the half-space $H \subset \mathbb{R}^3 \subset \mathbb{C}^2$ by rotating it around the axis ∂H and tracing out a surface in $\mathbb{C}^2$.
- Note that the meridian curve γ (the unit normal circle over a point in T^2) is a meridian of L in the half-space H.
- Now take α to be a Seifert pushoff of the knot K in H and β to be the orbit of a point on α under the rotation.
- We can perform the surgery topologically with respect to this framing and if L has a Lagrangian representative then the surgered manifold inherits a symplectic form. By Gromov's theorem it is still symplectomorphic to $\mathbb{C}^2$. In particular, π_1 of the surgered manifold is zero.

Lemma

If K is a nontrivial knot then π_1 of the surgered manifold is nontrivial.

Proof.

The fundamental group of the surgered manifold is isomorphic to the fundamental group of the 3-manifold obtained by k-Dehn surgery along the knot $K \subset H$. To see this, note that the surgery doesn't involve the curve β so the result of surgery is $\mathbb{R}^3_k(K) \times S^1 \cup \nu\{\text{axis}\}$ and use van Kampen. The fundamental group of the k-Dehn surgery on a nontrivial torus knot is known to be nontrivial for almost all values of k (the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen). □

Chern classes

Let E be a complex vector bundle (that is a vector bundle whose transition functions are in $GL(n, \mathbb{C})$) and let $\sigma_1, \ldots, \sigma_{n-k+1}$ be a generic collection of sections of E. The subset where $\sigma_1, \ldots, \sigma_{n-k+1}$ are complex linearly dependent we denote by S_{n-k+1} and we call the kth degeneracy locus. Its Poincaré dual is called the kth Chern class, $c_k(E)$.
This is a very functional definition and I'd love to tell you about where it comes from, but instead I'll refer you to Bott and Tu who do a very good job. The reason it's useful in symplectic geometry is that $GL(n, \mathbb{C})$ retracts onto $U(n)$ and so does $\mathrm{Sp}(2n)$: therefore we can turn a symplectic vector bundle (E, ω) into a complex vector bundle (E, J) by picking a fibrewise compatible almost complex structure J. There's a contractible set of such choices and the isomorphism class of (E, J) is a discrete thing so it can't jump when we change J in a continuous way.

We saw one calculation of $c_1(\mathcal{O}(-1))$ where $\mathcal{O}(-1)$ denotes the universal line bundle over $\mathbb{CP}^N$, where $\mathbb{CP}^N$ is the space of complex lines through the origin in $\mathbb{C}^N$. The bundle has fibre at a point $\pi \in \mathbb{CP}^N$ equal to the line π. A *meromorphic section* of this is given by intersecting lines with a hyperplane $H \in \mathbb{C}^{N+1}$ disjoint from the origin in $\mathbb{C}^{N+1}$. This section equals infinity along lines contained in the hyperplane parallel to H through 0. Denote by $h \subset \mathbb{CP}^N$ this subset. The reciprocal of the section vanishes along h but because we've taken reciprocal we have to reverse the orientation on the degeneracy locus. Therefore we get $c_1(\mathcal{O}(-1)) = -P.D.(h)$.

Now let's see some properties of Chern classes. We will not prove these but we'll motivate them. You can try and prove them in the ways indicated by being careful about transversality.

- If $f : X \to Y$ is a smooth map and $E \to Y$ is a complex vector bundle then $c_k(f^*E) = f^*c_k(E)$ (which makes sense because c_k is a cohomology class). To convince yourself this is true, consider a collection of sections σ_i of E and pull them back along f and perturb them to be generic.
- If $0 \to A \to E \to B \to 0$ is an exact sequence of vector bundles then

  $$(1+c_1(A)+c_2(A)+\cdots)(1+c_1(B)+c_2(B)+\cdots) = 1+c_1(E)+c_2(E)+\cdots$$

  In particular $c_1(A) + c_1(B) = c_1(E)$. This special case is easy to motivate: the bundle is split $E = A \oplus B$ and if a and b are respectively the ranks of A and B then $\sigma_1, \ldots, \sigma_a \in \Gamma(A)$, $\tau_1, \ldots, \tau_b \in \Gamma(B)$ (generic sections defining $c_1(A)$ and $c_1(B)$) give us $n + m$ sections $\sigma_i \oplus 0, 0 \oplus \tau_j$ whose degeneracy locus the union of the degeneracy loci of $\{\sigma_i\}$ and $\{\tau_j\}$.

Adjunction formula

For a Lagrangian submanifold the normal bundle is naturally identified with the cotangent bundle and this gives strong topological restrictions. For *symplectic submanifolds* we don't have such control, but Chern classes can give us some extra topological information.

If $\Sigma \subset X$ is a symplectic submanifold then its normal bundle $\nu\Sigma$ inherits a fibrewise symplectic form as we saw in Lecture III. This makes it a symplectic vector bundle and so it has Chern classes. Similarly the tangent bundles of Σ and of X admit fibrewise symplectic forms (by definition of a symplectic manifold) and their Chern classes are related by the *adjunction formula*:

$$c_1(T\Sigma) + c_1(\nu\Sigma) = c_1(TX|_\Sigma)$$

This is just the splitting formula for Chern classes applied to the bundle exact sequence

$$0 \to T\Sigma \to TX|_\Sigma \to \nu\Sigma = TX|_\Sigma / T\Sigma \to 0$$

Suppose Σ has real codimension 2.

- To calculate $c_1(TX|_\Sigma)$ take a collection of sections $\{\sigma_i\}$ of $TX|_\Sigma \to \Sigma$ and extend them generically to sections $\{\hat{\sigma}_i\}$ over the rest of X. The degeneracy locus of $\{\sigma_i\}$ is precisely the intersection of the degeneracy locus of $\{\hat{\sigma}_i\}$ with Σ, that is the value of $c_1(TX) \in H^2(X;\mathbb{Z})$ on $[\Sigma] \in H_2(X;\mathbb{Z})$.
- To calculate $c_1(\nu\Sigma)$, it suffices to calculate $[\Sigma]^2 \in H_{2n-4}$.

For $\dim \Sigma = 2$, $c_1(T\Sigma) = \chi(\Sigma) = 2 - 2g$.
For plane curves, $c_1(\mathbb{CP}^2) = 3H$ (H is the class of a line) and $c_1(\mathbb{CP}^2)[\Sigma]$ is called the degree of the curve Σ. Since $[\Sigma] = d[H]$, $[\Sigma] \cdot [\Sigma] = d^2$ and we obtain the degree-genus formula

$$3d = 2 - 2g + d^2$$

that is

$$g = \frac{(d-1)(d-2)}{2}$$

Exercise

Use the fact that $\mathbb{CP}^2$ can be covered with 3 complex coordinate patches to show that $c_1(\mathbb{CP}^2) = 3H$.

Here's another formula that might be appealing to physicists in the audience (if there are any left!). A symplectic Calabi-Yau manifold is a symplectic manifold X with $c_1(X) = 0 \in H^2(X; \mathbb{Z})$. The adjunction formula for curves in symplectic Calabi-Yau 4-manifolds (e.g. 4-torus, K3 surface) says

$$2g - 2 = [\Sigma]^2$$

so symplectic spheres have self-intersection -2, symplectic tori have self-intersection 0,...

The only other thing I wanted to recap was the Lagrangian Grassmannian, that is the space $L(n)$ of Lagrangian n-planes in the standard symplectic $\mathbb{R}^{2n}$. Recall that we showed $U(n)$ acts transitively on $L(n)$ with stabiliser $O(n)$, which we saw as follows. Let Λ be a Lagrangian n-plane, J a compatible complex structure and $e_1, \ldots, e_n$ an orthonormal basis for Λ. Then $e_1, Je_1, e_2, J_2, \ldots$ is an orthonormal symplectic (i.e. unitary) basis for $\mathbb{R}^{2n}$ and in these coordinates Λ is the standard Lagrangian $\mathbb{R}^n$.

This gives us a fibration

$$\begin{array}{ccc} O(n) & \longrightarrow & U(n) \\ & & \downarrow \\ & & L(n) \end{array}$$

We also have a fibration

$$\begin{array}{ccc} SU(n) & \longrightarrow & U(n) \\ & & \downarrow \det \\ & & S^1 \end{array}$$

and $SU(n)$ is simply connected so $\pi_1(U(n)) = \mathbb{Z}$ via the homotopy long exact sequence. Since $\pi_1(O(n)) = \mathbb{Z}/2$ for $n \geq 3$ and $\pi_0(O(n)) = \mathbb{Z}/2$, the homotopy long exact sequence of the first fibration gives

$$1 \to \mathbb{Z} \to \pi_1(L(n)) \to \mathbb{Z}/2 \to 1$$

To see that $\pi_1(L(n) = \mathbb{Z}$, consider that the path $\mathrm{diag}(e^{i2\pi\theta}) \in U(1)_{\mathrm{diag}} \subset U(n)$ takes the standard Lagrangian $\mathbb{R}^n$ to itself after time $\theta = 1/2$.