Lagrangian spheres in Del Pezzo surfaces

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Projective varieties

- A symplectic form is a closed, nondegenerate 2-form.

- Such a form is present on the complex projective space, the Fubini-Study form ω_{FS}, and pulls back to give a symplectic form ω_V any smooth subvariety $V \subset \mathbb{CP}^n$.

- Deforming the subvariety to V' you actually get (by an argument of Moser) a diffeomorphism $\phi : V \to V'$ such that $\omega_V = \phi^* \omega_{V'}$.

- So we can use the microscope of symplectic topology to examine projective varieties, introducing objects which are natural from the point of view of symplectic topology but maybe not from the pure algebra.

Today we'll be looking through the other end of this microscope to see what algebraic geometry suggests about the symplectic topology of some simple varieties.

- First I will explain what kind of objects the symplectic structure gives you,

- Then I will review how they arise in an algebro-geometry context by introducing the symplectic version of Picard-Lefschetz theory.

- Then I'll give some examples and theorems which illustrate what I mean about looking through the other end of the microscope.

Symplectic objects

What kind of objects does the symplectic form allow us to define?

- Symplectomorphisms: these are diffeomorphisms $\phi : V \to V$ such that $\phi^*\omega_V = \omega_V$ and they form an infinite-dimensional group

  $$\mathsf{Symp}(V, \omega_V) \subset \mathsf{Diff}(V)$$

  The component group $\pi_0(\mathsf{Symp}(V, \omega_V)$ is called the symplectic mapping class group and is often much bigger than its image in the smooth mapping class group $\pi_0(\mathsf{Diff}(V))$.

- Lagrangian submanifolds: these are submanifolds $L \subset V$ of real dimension $\dim_{\mathbb{C}} V$ on which $\omega|_L = 0$. The space of Lagrangian submanifolds $\mathcal{L}$ is an infinite-dimensional subspace of the space $\mathcal{S}$ of all submanifolds and its component set $\pi_0(\mathcal{L})$ is often much bigger than the image of its inclusion in $\pi_0(\mathcal{S})$ (two Lagrangian submanifolds can be isotopic as smooth submanifolds but not through a path of Lagrangian submanifolds).

Today we will restrict attention to Lagrangian spheres (we'll see why in a moment). The main object of today's talk is $\pi_0(\mathcal{L})$. We'll see an example (due to Seidel) where it's very large (in a way suggested by algebraic geometry) and we'll see an example (due to me) where it's very small (ditto).

Families

Both of these objects arise naturally in projective geometry.

- Suppose $V \to \mathcal{V} \to S$ is a family of smooth subvarieties of $\mathbb{CP}^n$ parametrised by some base S.

- Define a connection on this family by setting the horizontal spaces to be the symplectic orthogonal complements of the tangent spaces to the fibres (the symplectic orthogonal complement of a linear subspace $A \in T_p\mathcal{V}$ is the set of vectors v such that $\Omega(v, w) = 0$ for all $w \in T_pV$ where V is the fibre).

- Here the 2-form Ω on $\mathcal{V}$ is just the pullback of ω_{FS} along the (fibrewise injective) map $\mathcal{V} \to \mathbb{CP}^n$.

- Parallel transport with respect to this connection along a path $\gamma : [0,1] \to S$ gives diffeomorphims $\Phi_\gamma : V_{\gamma(0)} \to V_{\gamma(1)}$ which are symplectic i.e. $\Phi_\gamma^* \omega_{V_{\gamma(1)}} = \omega_{V_{\gamma(1)}}$. In particular, transport around loops gives a *symplectic monodromy representation*

$$\pi_1(S) \to \pi_0(\mathrm{Symp}(V, \omega_V))$$

- Now consider a *nodal degeneration* $\mathcal{V} \to \mathbb{C}$, that is a family of varieties such that all fibres away from 0 are smooth and the fibre at zero contains a single nodal singularity (locally modelled on $\{\sum x_i^2 = 0\} \subset \mathbb{C}^k$). We can define parallel transport along paths $\gamma : [0,1) \to \mathbb{C}$ where $\gamma(t) \to 0$ as $t \to 1$ and ask how the parallel transport diffeomorphisms behave in the limit.

Theorem 1 (Lefschetz, Seidel) *In the limit a Lagrangian sphere in V is crushed to a point: this is called the vanishing cycle. The symplectic monodromy around the unit circle is (isotopic to) a Dehn twist in this sphere.*

A Dehn twist in a Lagrangian sphere is a particular explicit symplectomorphism first written down by Arnold. It acts as the antipodal map on the Lagrangian sphere and is compactly supported near the sphere. We may as well define it to be the monodromy of a nodal degeneration. It is easy to see that the sphere should be Lagrangian: the parallel transport diffeomorphisms are symplectic and the limiting set is just a point (on which the symplectic form vanishes).

Setting

The setting can be summarised as follows:

- Projective varieties have symplectic structures,

- Families of varieties have symplectic parallel transport maps,

- The monodromy of a loop is a symplectomorphism,

- The vanishing cycle of a nodal degeneration of varieties can be represented as a Lagrangian sphere.

The questions we want to understand are:

Question 1 *Is the symplectic monodromy*

$$\pi_1(S) \to \pi_0(\mathsf{Symp}(V, \omega_V))$$

injective? Surjective?

Question 2 *Are there Lagrangian spheres which do not arise via nodal algebraic degenerations of V or does this construction capture them all?*

Today we're mostly interested in Question 2.

Example

The example to have in mind is the following.

- Take $k \leq 8$ (labelled) points in general position in $\mathbb{CP}^2$ (no three on a line, no six on a conic,...) and blow them up, using (a multiple of) the anticanonical linear system to embed into $\mathbb{CP}^n$. You get a projective variety which I'll denote by $\mathbb{D}_k$.

- You get a family S by varying the points (up to automorphisms $\mathbb{P}GL(3,\mathbb{C})$ of $\mathbb{CP}^2$).

- Since automorphisms act 4-transitively the first interesting case is $\mathbb{D}_5$ and already $\pi_1(S)$ is highly nontrivial (it's the group of pure 5-strand braids on S^2 modulo its centre).

- When you move the points out of general position (e.g. when two collide) you get a nodal degeneration (maybe after a base change) which gives a Lagrangian sphere.

- The simplest interesting case is $\mathbb{D}_2$, bringing the two points together along a line. There is exactly one way to do this, so we get a preferred Lagrangian sphere (up to isotopy through Lagrangian spheres), i.e. a preferred component in $\pi_0(\mathcal{L})$.

Theorem 2 (E.) *Fix a homology class* $A \in H_2(\mathbb{D}_k;\mathbb{Z})$ *for* $k = 2,3,4$ *which could potentially contain a Lagrangian sphere (i.e.* $A^2 = -2$ *and* $\omega \cdot A = 0$*). Then there is a unique Lagrangian sphere in that homology class up to isotopy through Lagrangian spheres. Put another way, the space* $\mathcal{L}(A)$ *of embedded Lagrangian spheres representing the homology class* A *is path-connected.*

This theorem builds on earlier work of Richard Hind who proved a similar statement for the quadric surface $S^2 \times S^2$. Let us contrast this theorem with the following observation of Seidel.

Theorem 3 (Seidel) *In* $\mathbb{D}_5$ *for a homology class* A *with* $A^2 = -2$*,* $\omega \cdot A = 0$ *we have* $|\pi_0(\mathcal{L}(A))| = \infty$*.*

To see this, we observe that the symplectic monodromy of our family S (from varying the points) is

1. generated by Dehn twists in the vanishing cycles we have constructed,

2. injective as a map $\pi_1(S) \to \pi_0(\mathrm{Symp}(\mathbb{D}_5))$,

3. infinite (i.e. $|\pi_1(S)| = \infty$).

To see (1) just observe that the space

$$\mathrm{Conf}_5^{\ell,g}\mathbb{CP}^2/\mathrm{Aut}$$

of configurations of five labelled (ℓ) points in general position (g) in $\mathbb{CP}^2$ can be partially compactified by allowing precisely the kinds of configurations in special position we used to construct our Lagrangian spheres and that the partial compactification has no π_1 left.

To see (2) is harder and requires pseudoholomorphic curves. The idea is to look at the smooth conic through the five points: this gives a map (actually a homeomorphism)

$$\kappa : S = \mathrm{Conf}_5^{\ell,g}\mathbb{CP}^2/\mathrm{Aut} \to \mathrm{Conf}_5^{\ell}\mathbb{CP}^1/\mathrm{Aut}$$

We can mimic this on the level of pseudoholomorphic curves: any compatible almost complex structure $J \in \mathcal{J}$ on $\mathbb{D}_5$ admits six embedded pseudoholomorphic curves $E_1, \ldots, E_5, C$ representing the five exceptional homology classes of the blow-up and the homology class of the proper transform of the conic. $E_i \cap C$ is a transverse intersection in a single point by positivity of intersections.

This gives a map

$$B\mathsf{Symp}_0(\mathbb{D}_5) = \mathcal{J}/\mathsf{Symp}_0(\mathbb{D}_5) \to \mathsf{Conf}^{\ell}_5\mathbb{CP}^1/\mathsf{Aut}$$

here the subscript 0 means "acting trivially on homology" and the object on the left is the classifying space for symplectic fibre bundles (because $\mathcal{J}$ is contractible and the group acts freely). Now the family $\mathcal{V} \to S$ from before is classified by a map $S \to B\mathsf{Symp}_0(\mathbb{D}_5)$ and the map κ from before factors (up to homotopy) through this map, proving (2) (in fact the classifying map is a homotopy equivalence [E.], another symplectic theorem suggested by algebraic geometry).

We therefore have *knotted Lagrangian spheres*. It's not too hard to check that the spheres in the above example are smoothly isotopic, but we have shown they're not Lagrangian isotopic (otherwise the Dehn twists they generate would be isotopic). We can visualise the Lagrangian spheres via κ by allowing pairs of points on $\mathbb{CP}^1$ to come together along a path (this corresponds to a degeneration where two points in $\mathbb{CP}^2$ come together). Non-isotopic paths should give non-isotopic Lagrangians.

We now discuss the proof of Theorem 2, which claims that in $\mathbb{D}_k$ ($k \leq 4$) the space $\mathcal{L}(A)$ is connected for each homology class A which could contain a Lagrangian 2-sphere. Lagrangian isotopy classifications come in two distinct flavours:

- Local: A neighbourhood of a Lagrangian L is known to be symplectomorphic to a neighbourhood of L inside its cotangent bundle (Weinstein) so the local question is the Arnold conjecture on nearby Lagrangians: is every (exact) Lagrangian Hamiltonian isotopic to the zero-section? In the case of S^2, any Lagrangian S^2 is exact and Hamiltonian isotopy is just Lagrangian isotopy, so the conjecture is settled by the following theorem of Hind:

  Theorem 4 (Hind) *Any Lagrangian sphere in T^*S^2 is Lagrangian isotopic to the zero-section.*

- Global: Can we use this local knowledge to prove path-connectedness of $\mathcal{L}(A)$ in a closed manifold? Hind actually already did this for $S^2 \times S^2$, where he had the luxury of foliations by pseudoholomorphic spheres. We will do it for $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$ where we still have the luxury of many pseudoholomorphic spheres, but not foliating.

Disjoining from divisors

The idea is to find a divisor $D_A \subset \mathbb{D}_k$ (depending only on the homology class) such that

- $\mathbb{D}_k \setminus D_A$ is symplectomorphic to a bounded subset of T^*S^2,
- Any Lagrangian sphere in the homology class A can be made disjoint from D_A by an isotopy through Lagrangian spheres.

To see the relevant divisors, note that $\mathbb{D}_k$ is biholomorphic to the blow-up of $S^2 \times S^2$ at $k-1$ points. Recall that $S^2 \times S^2$ splits as a neighbourhood of the diagonal (a holomorphic sphere) and a neighbourhood of the antidiagonal (a Lagrangian sphere). Blow up $k-1$ points on the diagonal and take D_A to be the total transform of the diagonal: its complement is still a neighbourhood of the antidiagonal!

To perform the disjunction, we will proceed in reverse.

- We will find a family $D_A(t)$ of configurations of symplectic spheres ("pseudoholomorphic divisors") such that $D_A(0) = D_A$ and $D_A(1)$ is disjoint from L.

- Now a standard theorem from symplectic topology due to Banyaga allows us to extend such an isotopy of spheres to a global family of symplectomorphisms ϕ_t, i.e. $\phi_t(D_A) = D_A(t)$.

- Then the family $L_t = \phi_t^{-1}(L)$ is an isotopy of Lagrangian spheres joining L to a Lagrangian sphere L_1 disjoint from D_A.

For simplicity I will show you how to disjoint L from a very special kind of divisor, namely a single curve E with $E^2 = -1$ and $c_1(E) = 1$ (an exceptional curve). This actually suffices for $\mathbb{D}_4$. The general case is not much harder.

Theorem 5 *For any monotone symplectic 4-manifold X (think $\mathbb{D}_2$, $\mathbb{D}_2$, $\mathbb{D}_4$), any symplectic exceptional sphere E can be disjoined from a Lagrangian sphere L if $E \cdot L = 0$ in homology.*

Aside

This theorem has been considerably generalised by Li and Wu in a recent preprint. The monotonicity assumption is replaced by Kodaira dimension $-\infty$, E by any symplectic sphere and disjunction by (geometric intersection with L equal to homological intersection). This has some great consequences and is probably the final step in the story of Lagrangian spheres in Del Pezzo surfaces. The ideas are very similar to the ones I will sketch here.

Neck-stretching

- The idea is to find a family of almost complex structures J_t on $\mathbb{D}_k$ and to let $E(t)$ be the unique J_t-holomorphic sphere representing the homology class E.

- The fact that there is a unique embedded J_t-holomorphic sphere representing an exceptional class in a monotone 4-manifold is due to the positivity of intersections between pseudoholomorphic curves in four dimensions (not obvious, but easy).

- The family J_t is obtained by stretching the neck around L. You take some fixed, standard complex structure on a neighbourhood of L (which, recall, looks like a subset of T^*S^2) and then stretch it out in a translation-invariant way in a neck around the boundary of the neighbourhood.

- "Stretch" means that the neck becomes longer and longer relative to the metric obtained using ω and J_t.

Such degenerations of almost complex structures are considered in symplectic field theory and the limits of J_t-holomorphic curves as $t \to \infty$ have been studied intensively since the work of Hofer. The compactness theorem of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder tells us that in the limit the curve E_t breaks up into various "finite-energy curves" with different levels:

- one in T^*L,
- one in the complement of L,
- possibly some intermediate cylindrical levels which look like an infinite neck.

All of these levels are potentially noncompact curves but the “finite-energy” condition I alluded to means that they are well behaved asymptotically: they asymptote to Reeb orbits at infinity. Reeb orbits in our case are just (possibly multiple covers of) geodesics in the cotangent bundle (with respect to the round metric on S^2).

The hope is to show that E_∞ has no component in T^*L: then by the nature of the convergence used in BEHWZ (Gromov-Hofer convergence), E_t must be disjoint from L for sufficiently large t.

When E has $c_1(E) = 1$ it is easy to show that the $X \setminus L$-level of E_∞ is connected and not multiply-covered: one can define a first Chern class relative to the boundary and different components contribute positively to c_1 by monotonicity of X (positive area implies positive Chern class), but 1 is the smallest integer.

One can also show that for generic choice of family J_t the asymptotic Reeb orbits of $E_\infty \cap X \setminus L$ are distinct and simple (i.e. no multiple covers). This is a combination of transversality results none of which are harder than transversality for simple closed holomorphic curves in standard Gromov-Witten theory:

- multiply-covered Reeb orbits give negative expected dimension and the underlying curve is simple;

- distinctness can be achieved by looking at the asymptotic evaluation map to the space of Reeb orbits and ensuring transversality of this map with the multidiagonal.

Now suppose that $E_\infty \cap T^*S^2$ is nonempty. It must consist of finite energy planes with simple distinct Reeb orbits (planes only in order to ensure that E_∞ has genus 0). But such planes are easily classified: think of T^*S^2 as a neighbourhood of the antidiagonal in $S^2 \times S^2$. This is the same as collapsing the Reeb orbits to points (which now form the diagonal). The finite energy curves compactify to holomorphic spheres in $S^2 \times S^2$ which intersect the diagonal once transversely and these come in two families: $\alpha = S^2 \times \{\star\}$ and $\beta = \{\star\} \times S^2$ which intersect the antidiagonal L homologically as $\alpha \cdot L = 1$ and $\beta \cdot L = -1$. Since $E \cdot L = 0$ we need the same number of α and β planes to cancel out the homological intersection. So if $E_\infty \cap T^*L \neq \emptyset$ then there is at least one α plane and one β plane with different asymptotics. But then they intersect (transversely) inside T^*S^2 and E_∞ is not embedded. The nature of Gromov-Hofer convergence implies that E_t is not embedded for large t and this contradicts the fact that all E_t are embedded.

Summary

To sum up, we have shown how to disjoin a Lagrangian sphere from an exceptional sphere. The same methods work to disjoin Lagrangian spheres from certain well-chosen divisors D_A in $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$ whose complements are symplectomorphic to T^*S^2 and where we can apply Hind's isotopy uniqueness (which is proved by similar, more involved neck-stretching arguments) to show that the space of Lagrangians in the homology class A ($\mathcal{L}(A)$) is connected. In particular, all the Lagrangian spheres arise as vanishing cycles of nodal degenerations.

In $\mathbb{D}_5$ there is no such divisor and indeed we saw that knotting (disconnectedness of $\mathcal{L}(A)$) can occur. However, the recent preprint of Li and Wu (coupled with some work of mine on the symplectic mapping class group of $\mathbb{D}_5$) actually allows one to prove that all the Lagrangian spheres arise as vanishing cycles of nodal degenerations in this case as well and I'm optimistic that the question will soon be answered affirmatively for all blow-ups of $\mathbb{CP}^2$ using their work (and some input on the mapping class group).

It is interesting to note that Corti-Smith have examples in dimension six of Lagrangian 3-spheres which do not arise as vanishing cycles of algebraic degenerations, and that the methods I have outlined are absolutely confined to four dimensions.