Isotopy of Lagrangian Submanifolds

1. Reasons to study Lagrangian spheres (as opposed to other possible topologies):
   * They are global objects, e.g. Gromov ⇒ there are none in a Darboux chart.
   * They are simply-connected, so Lagrangian isotopies extend to global Hamiltonian symplectomorphisms.
   * They give rise to non-trivial symplectomorphisms (Dehn twists).
   * They arise in algebraic geometry as vanishing cycles.

   } Let's examine these properties further.

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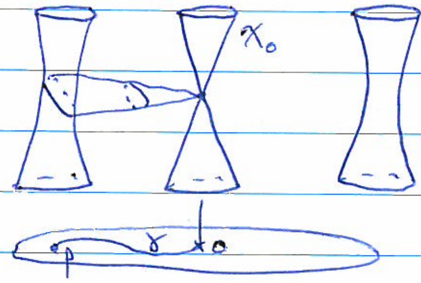
2. $\mathcal{X} \to S$ family of projective varieties where X_0 has a single node.



The Fubini-Study form ω_s on X_s varies smoothly & gives us a symplectic connection $Hor = (T_{vert} X_s)^{\omega_s \perp} \subseteq T\mathcal{X}$. ⇝ This gives us a good notion of symplectic parallel transport (away from 0).

Lemma - Def^{n}: The vanishing cycle of a path γ from p to 0 is the subset of X_p which collapses to the node of X_0 under parallel transport. This is a smooth, embedded Lagrangian sphere L. Monodromy of a loop around 0 is the Dehn twist in L, a symplectomorphism.

* Can we classify embedded Lagrangian spheres up to (Lagrangian, embedded) isotopy?
* Can we understand how much of the symplectic mapping class group is generated by Dehn twists?

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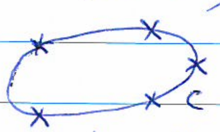
3. Seidel has thought about these questions a lot. Here is a pretty picture which emerges from his work.

$\mathbb{D}_5 := 5$-point blow-up of $\mathbb{CP}^2$ $\quad [\omega] = 3[H] - \sum_{i=1}^{5} [E_i]$

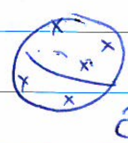
$\boxed{\text{Construction of knotted Lagrangians, I}}$

$$\begin{array}{c} \mathcal{X}^{univ} \\ \downarrow \\ Conf^{gen}_{ord}(5, \mathbb{CP}^2)/PSL(3,\mathbb{C}) \xrightarrow{\cong} Conf_{ord}(5, \mathbb{CP}^1)/PSL(2,\mathbb{C}) \end{array}$$



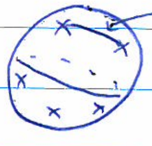
! conic through 5 points

$\longmapsto$



config. of intersections of proper transform of C with $E_1, \dots, E_5$.

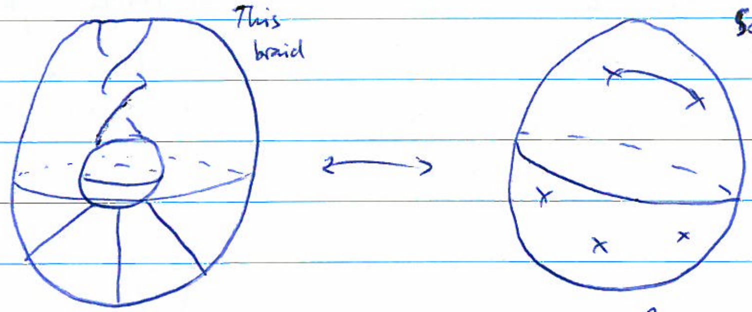
Lagr. sphere L $\longleftrightarrow$



γ a vanishing path (degeneration becomes nodal after base change)

4. $\pi_1\left[\frac{\mathrm{Conf}_{ord}(5,\mathbb{CP}^1)}{\mathbb{P}SL(2,\mathbb{C})}\right] = Br(5,S^2)/\langle \text{full twist}\rangle$

$\boxed{\text{Construction of knotted Lagrangians, II}}$



Monodromy of $X^{univ} \rightsquigarrow \Theta : \pi_1\left[\frac{\mathrm{Conf}_{ord}(5,\mathbb{CP}^1)}{\mathbb{P}SL(2,\mathbb{C})}\right] \longrightarrow \pi_0(\mathrm{Symp}_0 D_5)$

Th$^{\underline{m}}$: (Seidel) This map Θ is injective.

Proof: This is just the statement that 4-dim$^{\underline{l}}$ automatic transversality for -1-curves allows one to mimic the map $\mathrm{Conf}^{ge}_{ord}(5,\mathbb{CP}^2)/G \to \mathrm{Conf}_{ord}(5,\mathbb{CP}^1)/\mathbb{P}SL(2,\mathbb{C})$ with pseudohol. curves & get a htpy right inverse to the map

$$\frac{\mathrm{Conf}^{gen}_{ord}(5,\mathbb{CP}^2)}{G} \to B\mathrm{Symp}_0 D_5 \left(\xrightarrow{\downarrow} \mathrm{Conf}_{ord}(5,\mathbb{CP}^1)/\mathbb{P}SL(2,\mathbb{C}).\right) \quad \square$$

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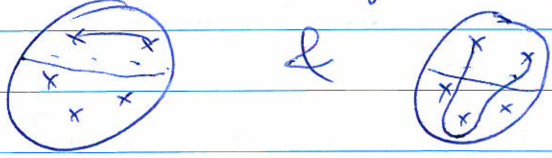
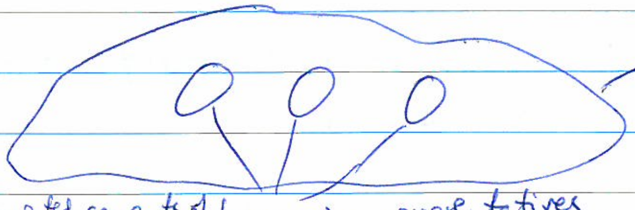
5. Corollary: The D.T.s in the Lagrangian spheres constructed as vanishing cycles for



are not isotopic.

In particular the spheres are not isotopic as Lagr. spheres.

They are, however, isotopic as smooth submanifolds, so we see the following phenomenon:



$\boxed{\text{Lagrangian knottedness}}$

Th$^{\underline{m}}$: (Seidel) Locally, this always occurs. That is, in ~~any~~ a nbhd of an A_n-config. of $(n\geq 2)$ Lagrangian spheres one can construct symplectically knotted (smoothly unknotted) Lagrangians by Dehn twisting them around one another.

6. For $\mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4$, $PSL(3,\mathbb{C})$ acts transitively on the blow-up points so one cannot construct a non-trivial moduli space whose π_1 gives interesting monodromies. Maybe there is hope to classify Lagrangian spheres?

Theorem A (Hind) * A Lagrangian sphere in $(S^2\times S^2, \omega_{std}\oplus\omega_{std})$ is Lagrangian isotopic to the antidiagonal $\bar{\Delta}=\{(x,-x) : x\in S^2, - = \text{antipodal map}\}$.
* A Lagr. sphere in T^*S^2 is isotopic to the 0-section.

Theorem B (E.) * There is precisely one Lagrangian sphere up to isotopy in any homology class with $\omega\cdot L=0$, $L^2=-2$ in the monotone Del Pezzo surfaces $\mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4$.
* In any of the D.P. surfaces $\mathbb{D}_{k<8}$ there is precisely one smooth isotopy class of spheres admitting Lagrangian representatives in each such homology class.

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7. Comment: Both of these theorems use symplectic field theory. Theorem A uses it to construct a pair of holomorphic foliations in the homology classes $S^2\times\cdot$, $\cdot\times S^2$ ~~which~~ whose leaves intersect L exactly once transversely. Then a clever Moser argument constructs a symplectomorphism which takes L to $\bar{\Delta}$. But $\pi_0(\mathrm{Symp}_0\, S^2\times S^2)=\{1\}$.
Theorem B uses SFT to construct an isotopy which shifts L off a divisor of genus 0 holo^c curves whose complement is symplectomorphic to a nbhd of 0 in T^*S^2, then applies theorem A.

I will illustrate the SFT-disjoining idea by proving the following general theorem which I don't believe is written down anywhere:

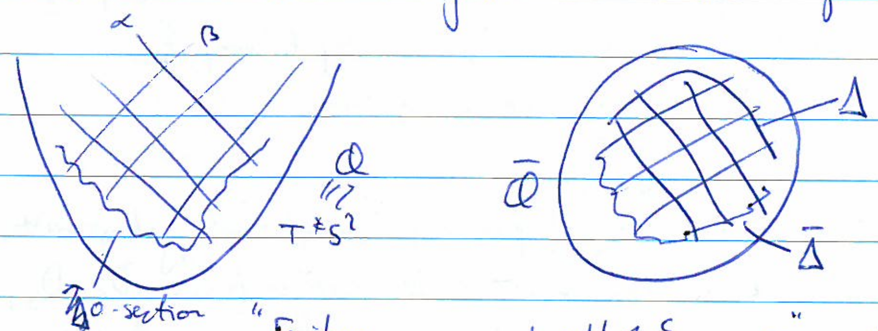
$$\boxed{\text{Theorem: Let } X \text{ be a symplectic 4-manifold, } L \text{ a Lagr. sphere, } E \text{ a holomorphic } -1\text{-curve with } c_1(E)=1. \text{ Then if } [L]\cdot[E]=0 \text{ there is an isotopy which moves } L \text{ off } E.}$$

You might like to think why this + theorem A implies the smooth unknottedness result in theorem B.

8. Review of T^*S^2 – Can think of this as the affine quadric $Q \subseteq \mathbb{C}^3$.
– Real part $Q_{\mathbb{R}} = S^2$ zero-section.
– Compactification $\bar{Q} \cong S^2 \times S^2$

Compactifying locus $= \Delta$ (diagonal)
real S^2 $= \bar{\Delta}$ (antidiagonal)

The α & β rulings $S^2 \times \cdot$, $\cdot \times S^2$ restrict to give holomorphic planes in Q.



"Finite energy punctured holo^c curves" are the right thing to replace closed holo^c curves in the non-compact setting.

* Asymptotically they look like cylinders of orbits of the Reeb v.f. at ∞.
* In our case the Reeb field is just geodesic flow on $ST^*S^2 \cong \mathbb{RP}^3$: all orbits are closed.
* $\mathbb{RP}^3$ / Reeb foliation $= \Delta$ (Hopf fibration)

i.e. to understand asymptotics of a punctured holo^c curve in Q it suffices to understand the intersection of its compactification with Δ.

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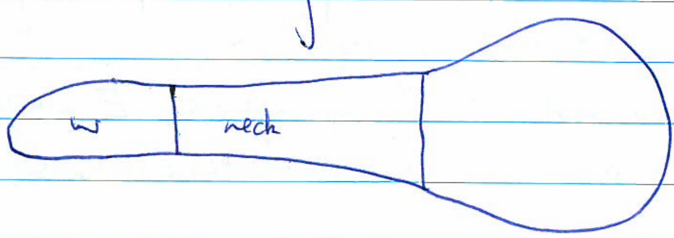
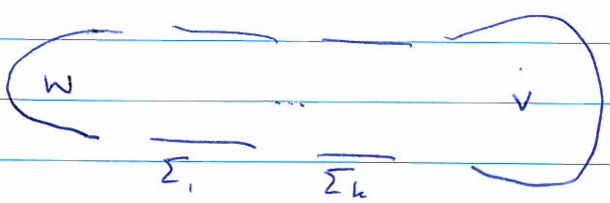
9. (Weinstein) L has a nbhd W like $U \subseteq T^*S^2$. Fix a standard almost complex structure on W & extend arbitrarily over $V = X \setminus W \rightsquigarrow J$.
"Stretch the neck" along $\mathbb{RP}^3 = \partial W \rightsquigarrow J_t$



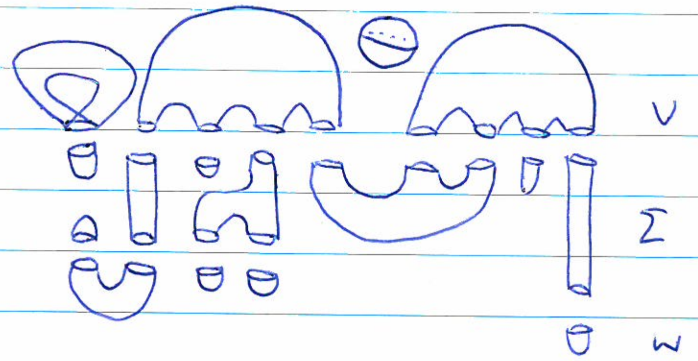
Since E is a -1-curve in a 4-manifold with $c_1(E) = 1$ there's a unique embedded J_t-holo^c curve homologous to E (by adjunction & automatic $\pitchfork$). $E(J_t)$ gets stretched.

Th^m As $t \to \infty$ one can find a subsequence $E(J_{t_i})$ which Gromov–Hofer converge to a broken holomorphic curve:

with components in W, V and in



10. <u>A possible broken curve:</u>



We must whittle this down to just  V.

For then Gromov-Hofer cvg^ce $\Rightarrow$ $E(J_T)$ is disjoint from L for some large T.

$E(J_t)\big|_{t=0}^{T}$ is then an isotopy of sympl. submflds, which extends (Auroux-Banyaga) to a global isotopy $\phi_t(E) = E(J_t)$. Then $\phi_t^{-1}(L)\big|_{t=0}^{T}$ is a disjoining isotopy of L from E.

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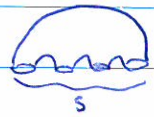
11. a) <u>V-part:</u> Can define a first Chern class for punctured hol^c curves relative to a fixed trivialisation of the cpx determinant line near ∂W.

$$c_1(E) = c_1(E^V) + \sum_{i=1}^{k} c_1(E^{\Sigma_i}) + c_1(E^W)$$

$c_1(E) = 1$, $c_1(E^{\Sigma_i}) = 0$, $c_1(E^W) = 0$ — T^*S^2 is anticanonical complement.

$\therefore$ $c_1(E^V)$ is minimal, which $\Rightarrow$ E^V is connected & simple.

$$\text{Edim}\,\mathcal{M} = 2\left(s - 1 + c_1(E^V)\right) - \sum_{i=1}^{s} 2\,\text{cov}_i$$
$$= 2s - 2\sum \text{cov}_i \le 0$$



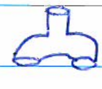
equality $\Leftrightarrow$ $\text{cov}_i = 1$.

Generic choice of J gives

$\dim \mathcal{M} = \text{Edim}\,\mathcal{M}$ & $\quad ev_{\text{Reeb}} : \mathcal{M} \longrightarrow (\text{Reeb } S^2)^s$

dim 0 $\qquad \pitchfork$ multidiagonal

$\Rightarrow$ distinct asymptotics.  ×

b) <u>Symplectisation part:</u> Energy $= \int u^* d\lambda = \sum_{i=1}^{s^+} \text{cov}_i^+ - \sum_{i=1}^{s} \text{cov}_i \ge 0$  ×

genus 0 $\Rightarrow \sum \text{cov}_i^+ \le 1$. At least one $-^{ve}$ end $\because$ simple Reeb orbit is htpclly non-triv.

$\Rightarrow$ all cylinders.

c) Only f.e. punctured planes in T^*S^2 are the α & β-planes. If one occurs, must have other to cancel homological intersection with L. But they're asymptotic to different Reeb orbits, so must intersect inside T^*S^2 $\Rightarrow$ $E(J_t)$ not embedded for large t. ※

$\therefore$ <u>W-part is empty.</u>

$\square$