Mappa Moduli

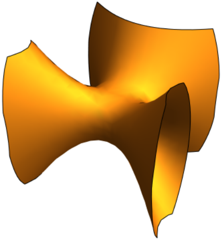
Singularities, Surfaces, and Symplectic Topology

Jonny Evans

31st October 2018

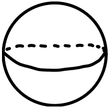
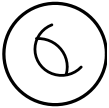
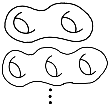
Varieties

A *variety* is a space cut out by polynomial equations.

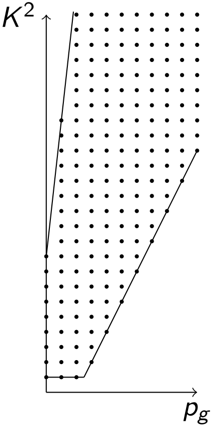


$$x^2y = y^3 + z^2 + 1$$

Classifying varieties

	$-\infty$	0	1	
Curves				
	$-\infty$	0	1	2
Surfaces	Rational/ruled	Calabi-Yau	Elliptic	General type

Surfaces of general type

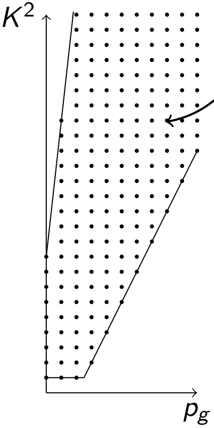


- Roughly classified by p_g, K^2.

Surfaces of general type

$\mathcal{M}_{p_g,K^2}$

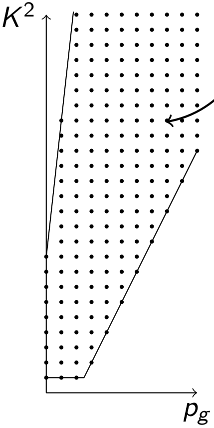
Mappa Mundi c.1300, Wikipedia



- Roughly classified by p_g, K^2.
- For each dot in the diagram we get a *moduli space* $\mathcal{M}_{p_g,K^2}$...

Surfaces of general type

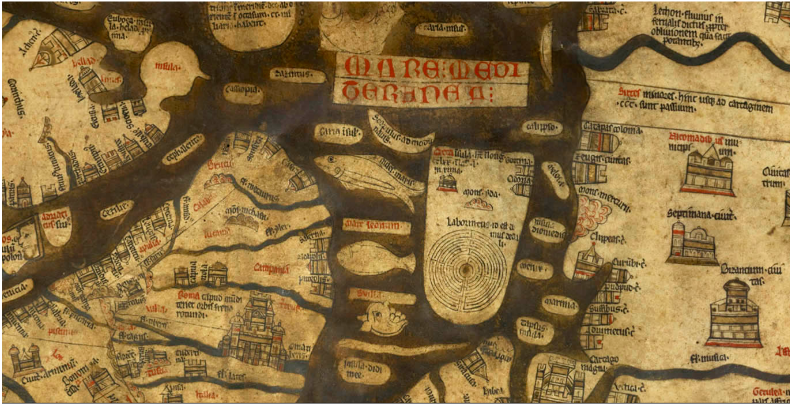
Mappa Mundi c.1300, Wikipedia



- Roughly classified by p_g, K^2.
- For each dot in the diagram we get a *moduli space* $\mathcal{M}_{p_g,K^2}$...
- ...about which we know very little.

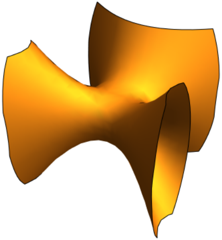
Some things we don't know about $\mathcal{M}_{p_g,K^2}$

- Dimension.
- Number of components.
- What happens at the boundary?



At the edge of moduli space

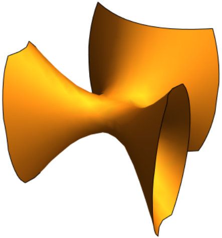
As the equations vary, the variety moves around in moduli space...



$$x^2y = y^3 + z^2 + 1$$

At the edge of moduli space

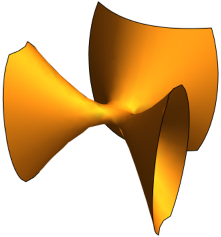
As the equations vary, the variety moves around in moduli space...



$$x^2y = y^3 + z^2 + 0.5$$

At the edge of moduli space

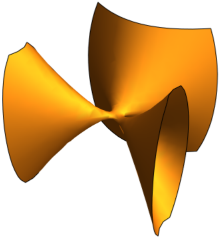
As the equations vary, the variety moves around in moduli space...



$$x^2y = y^3 + z^2 + 0.1$$

At the edge of moduli space

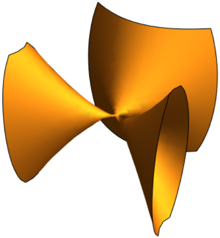
As the equations vary, the variety moves around in moduli space...



$$x^2y = y^3 + z^2 + 0.01$$

At the edge of moduli space

...and may develop singularities when it reaches the boundary.



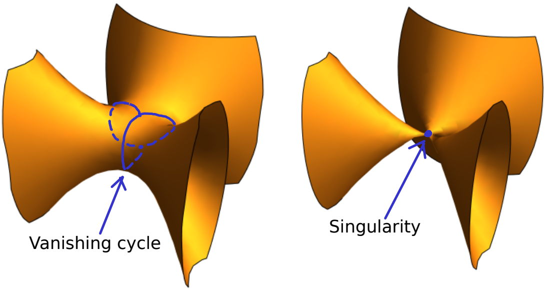
$$x^2y = y^3 + z^2 + 0$$

Question

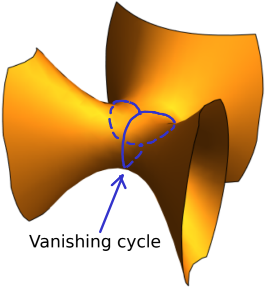
What singularities appear for varieties at the boundary of $\mathcal{M}_{p_g,K^2}$?

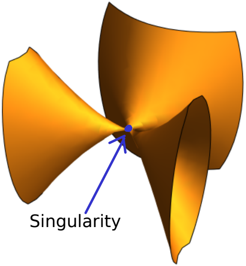
Past work on this problem

Vanishing cycles



Vanishing cycles





Strategy

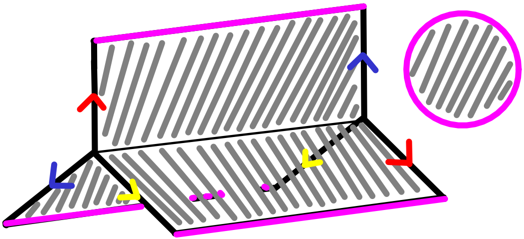
If smooth variety does not contain the required vanishing cycle then the corresponding singularity cannot appear.

Example: Wahl singularities

Fix $p \geq q \geq 1$ with $\gcd(p, q) = 1$

$$W_{p,q} = \mathbb{C}^2/\sim, \qquad (x, y) \sim (\mu x, \mu^{pq-1} y),\ \mu^{p^2} = 1.$$

- Occur frequently at the boundary of $\mathcal{M}_{p_g, K^2}$.
- The vanishing cycle is a *Lagrangian pinwheel* $L_{p,q}$.



Wahl singularities and $\mathbf{CP}^2$

Theorem *(Evans–Smith 2018, Geometry & Topology)*

If $L_{p_1,q_1}, \ldots, L_{p_n,q_n}$ are pairwise disjoint Lagrangian pinwheels in $\mathbf{CP}^2$ then $n \leq 3$. If $n = 3$ then

$$p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.$$

Wahl singularities and $\mathbf{CP}^2$

Theorem *(Evans–Smith 2018, Geometry & Topology)*

If $L_{p_1,q_1}, \ldots, L_{p_n,q_n}$ are pairwise disjoint Lagrangian pinwheels in $\mathbf{CP}^2$ then $n \leq 3$. If $n = 3$ then

$$p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.$$

- Corresponding constraint on singularities was proved using algebro-geometric methods earlier (Hacking–Prokhorov 2005).

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes.

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes. No explicit bound.

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes. No explicit bound.
- Alexeev-Mori 1995: Explicit bound.

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes. No explicit bound.
- Alexeev-Mori 1995: Explicit bound. *"It has to be admitted, however, that these bounds are quite high."*

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes. No explicit bound.
- Alexeev-Mori 1995: Explicit bound. *"It has to be admitted, however, that these bounds are quite high."*
- Lee 1999: $\ell \leq 400(K^2)^4$.

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes.
- Alexeev-Mori 1995: Explicit bound.
- Lee 1999: $\ell \leq 400(K^2)^4$.

No explicit bound.

"It has to be admitted, however, that these bounds are quite high."

$$\frac{p^2}{pq-1} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_\ell}}}$$

Main result

Question (Kollár, Shepherd-Barron 1988)

Fix p_g, K^2. Is there a bound on p such that the Wahl singularity $W_{p,q}$ can appear at the boundary of $\mathcal{M}_{p_g,K^2}$?

- Alexeev 1994: Yes. — No explicit bound.
- Alexeev-Mori 1995: Explicit bound. — *"It has to be admitted, however, that these bounds are quite high."*
- Lee 1999: $\ell \leq 400(K^2)^4$.

$$\frac{p^2}{pq-1} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_\ell}}}$$

Theorem *(Evans–Smith 2017)*

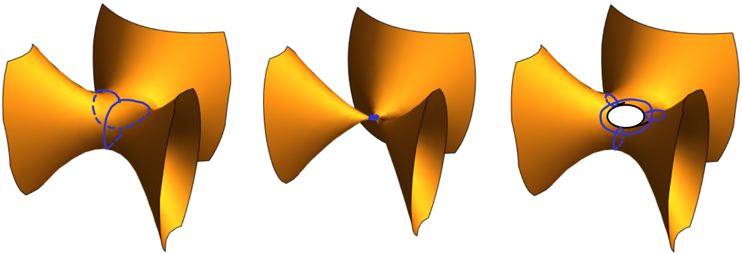
$$\text{If } p_g > 0 \text{ then } \ell \leq 4K^2 + 7.$$

Proof uses pseudoholomorphic curves and Seiberg-Witten theory.

Ongoing projects
exploring moduli space

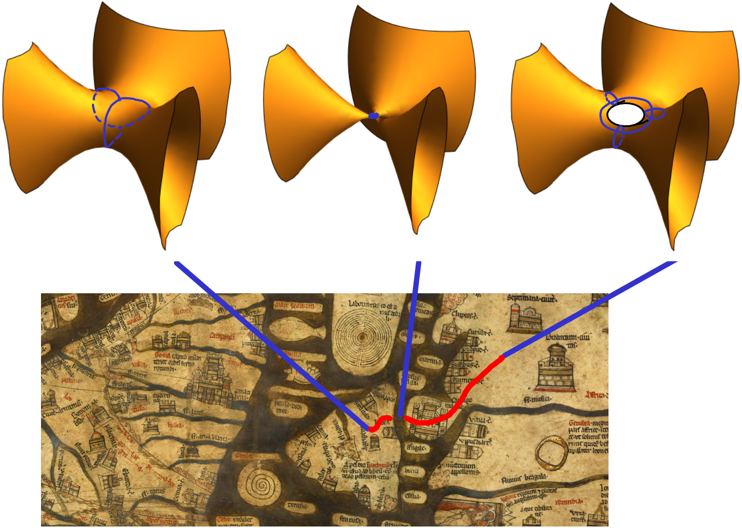
Topological transitions (joint with Giancarlo Urzúa)

Some singularities can be smoothed in multiple ways...



Topological transitions (joint with Giancarlo Urzúa)

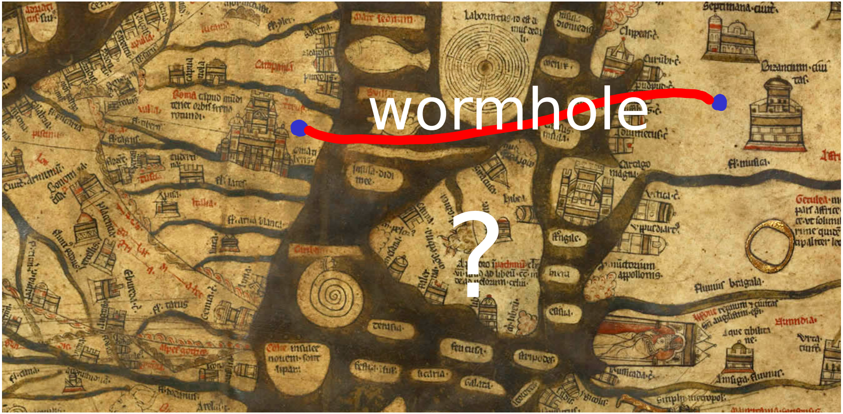
...giving *"wormholes"* between distant points of $\mathcal{M}_{p_g,K^2}$.



Topological transitions (joint with Giancarlo Urzúa)

Question

When do wormholes connect different components of $\mathcal{M}_{p_g,K^2}$?

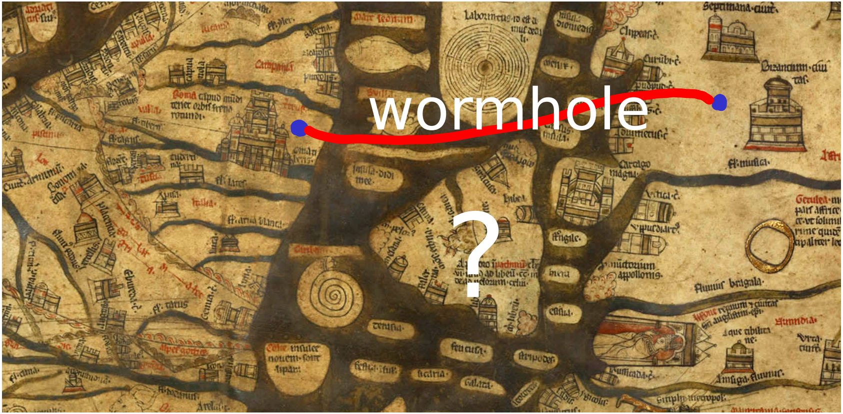


Topological transitions (joint with Giancarlo Urzúa)

Question

When do wormholes connect different components of $\mathcal{M}_{p_g, K^2}$?

- *Strategy:* If the topology changes then the wormhole connects different components.

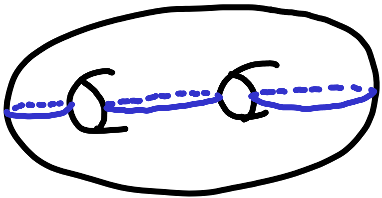


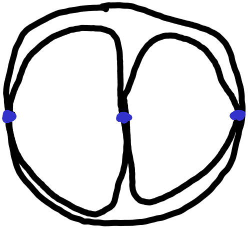
Large complex structure limits

Wahl singularities are (in a sense) the mildest singularities at the boundary of the moduli space.

Question

What are the *most singular* surfaces that occur?



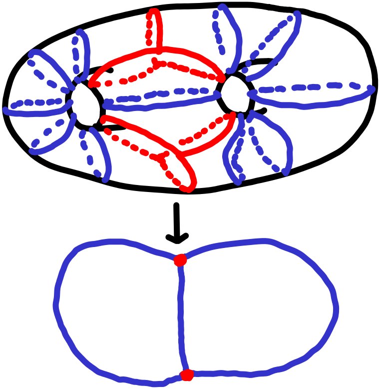


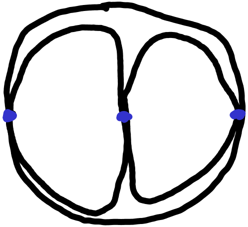
Large complex structure limits

Wahl singularities are (in a sense) the mildest singularities at the boundary of the moduli space.

Question

What are the *most singular* surfaces that occur?



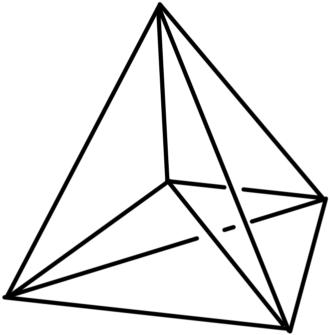


Large complex structure limits

Question

What are the *most singular* surfaces that occur?

- Does this yield Lagrangian torus fibrations on surfaces of general type?
- Can we use this to understand their geometry?



- Base of Lagrangian torus fibration on a quintic surface.
- There are four "holes" because $p_g = 4$.

Summary

- Surfaces of general type form a moduli space $\mathcal{M}_{p_g,K^2}$.
- Constraints on Wahl singularities:
  - for degenerations of $\mathbf{CP}^2$ (Markov equation),
  - for surfaces at the boundary of moduli space ($\ell \leq 4K^2 + 7$).
- Can we use topological transitions to find new components of moduli space?
- Can we use large complex structure limits to find Lagrangian torus fibrations on surfaces of general type?