# LUTTINGER SURGERY ON LAGRANGIAN TORI

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## 1. Introduction

The focus of this talk is the rigidity phenomena displayed by Lagrangian tori in the standard symplectic vector space $\mathbb{R}^4$. Let me recall:

- The standard symplectic vector space is

$$(\mathbb{R}^{2n}, \omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n)$$

- A Lagrangian submanifold $L$ is an $n$-dimensional embedded submanifold whose tangent spaces are $\omega$-isotropic. (We'll only use orientable ones).

Caveat: All submanifolds are assumed orientable and parametrised, i.e. we actually have a map $\iota\colon L \to \mathbb{C}^2$.

**Example 1.** $\mathbb{R}^n \subset \mathbb{R}^{2n}$ *given by*

$$\{x_1 = \cdots = x_n = 0\}$$

*is a Lagrangian submanifold. We're more interested in compact examples like the Lagrangian $n$-torus*

$$\{x_1^2 + y_1^2 = 1, \ldots, x_n^2 + y_n^2 = 1\}$$

We can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z_k = x_k + iy_k$. We have that $iTL \perp TL$, which means that multiplication gives an isomorphism $TL \cong \nu L$ between the tangent and normal bundles of $L$.

In $\mathbb{C}^2$ the normal bundle is trivial: it's classified by its Euler class which counts the homological self-intersection of $L$, in $\mathbb{C}^2$ this is zero since any submanifold can be displaced. Therefore the tangent bundle of $L$ must also be trivial and hence (remember we're assuming $L$ is orientable) $L$ is a 2-torus.

*Luttinger surgery* is a surgery operation which takes a Lagrangian torus in a symplectic manifold and gives back a new symplectic manifold with a Lagrangian torus. It is analogous to Dehn surgery in 3-dimensional topology.

## 2. Review of Dehn surgery

Given a knot in a 3-manifold $\iota\colon K \hookrightarrow M^3$ let $\nu K$ denote the normal bundle of $K$ and $T$ denote its boundary $\partial \nu K$. The exponential map (for some metric) gives an embedding of some disc subbundle of $\nu K$ into $M$ so I'll blur the distinction between normal bundle and tubular neighbourhood.

The normal bundle is trivial, but not in a canonical way. To define Dehn surgery we need an explicit trivialisation. We call this a choice of framing $K \times D^2 \cong \nu K$.

2.1. **Framings.** A framing is a diffeomorphism $K \times D^2 \to \nu K$ which commutes with the projections to $K$. In particular any two are related by a gauge transformation of the trivial $D^2$-bundle.

**Lemma 2.** *Gauge transformations (and hence framings) up to homotopy are in bijection with homology classes $H^1(K; Z) \cong \mathbb{Z}$.*

*Proof.* A gauge transformation of the trivial bundle is just a map from $K$ to $\mathrm{Diff}(S^1)$ which is homotopy equivalent to $S^1$ itself. Homotopy classes of maps $K \to S^1$ biject with $H^1(K; Z)$. Equivalently, we can canonically-up-to-homotopy identify the meridian $\gamma \subset T$ which is the boundary of a normal fibre $D^2$. The framing is a choice of longitude, that is a curve in $T$ which projects with degree 1 down to $K$. There are $\mathbb{Z}$ possibilities depending on how many times this longitude wraps around the meridian. □

Note that a knot in $\mathbb{R}^3$ has a canonical framing called the Seifert framing. To define it, since any knot is nullhomologous in $\mathbb{R}^3$ it bounds an orientable (so-called 'Seifert') surface and we push the knot $K$ off itself inside this surface. This gives a choice of longitude. Alternatively we can specify the longitude as a homology class in a way which makes it clear that the framing doesn't depend on the choice of Seifert surface. Namely, consider the Mayer-Vietoris sequence associated to the decomposition $M = (M \setminus \nu K) \cup \nu K$:

$$0 \to H_1(T; \mathbb{Z}) \to H_1(M \setminus \nu K; \mathbb{Z}) \oplus H_1(\nu K; \mathbb{Z}) \to 0$$

and let $[\lambda]$ be the longitude which is the preimage of $(0, 1)$ under this map. Think about it! This means that $\lambda$ is nullhomologous in the complement (still bounds a Seifert surface) and projects with degree 1 to $K$.

2.2. **Surgery.** Now given a framed knot we define the Dehn surgery with coefficient $1/k$ to be

$$(M \setminus \nu K) \cup_\phi \nu K$$

where $\phi$ is the gauge transformation (diffeomorphism of $T$) which sends the longitude $\lambda$ of the chosen framing to $\lambda + k\gamma$. The notation means that we cut out the tubular neighbourhood and then glue it back in, identifying $x \in \partial(M \setminus \nu K)$ with $\phi(x) \in \partial(\nu K)$.

The great thing about Dehn surgery is that it allows you to construct any 3-manifold (given a 3-manifold $M$ there exists a sequence of framed knots in $S^3$ on which you can perform Dehn surgery to get $M$).

## 3. Luttinger surgery

To do something similar for Lagrangian tori instead of knots we first need to discuss framings. For smooth embeddings $T^2 \hookrightarrow \mathbb{C}^2$ the normal bundle is trivial and to specify a framing it suffices to specify two numbers (an element of $H^1(T^2; \mathbb{Z})$). In terms of the Mayer-Vietoris sequence one sees

$$H_1(T = \partial \nu T^2; \mathbb{Z}) \overset{F}{\cong} H_1(\mathbb{C}^2 \setminus T^2; \mathbb{Z}) \oplus H_1(T^2; \mathbb{Z})$$

and if $a$ and $b$ denote $(1, 0)$ and $(0, 1)$ in $H_2(T^2; \mathbb{Z})$ then the framing is just a specification of $\alpha = F^{-1}((A, (1, 0)))$ and $\beta = F^{-1}((B, (0, 1)))$ in the cohomology of $T$. Again, $A = B = 0$ gives a canonical Seifert framing. But Lagrangian tori come with another canonical framing: the Weinstein framing.

3.1. **Weinstein framing.** Earlier we said that there is an isomorphism $TL \cong \nu L$. In fact

**Theorem 3** (Weinstein)**.** *There exists a neighbourhood $L \subset U \subset \mathbb{C}^2$, a neighbourhood $L \subset U' \subset T^*L$ and a symplectomorphism $U' \cong U$ taking $L$ identically to $L$.*

Here the cotangent bundle $T^*L$ is equipped with its canonical symplectic structure. As I'm in France, I'll be conceptual. Points in $T^*L$ parametrise 1-forms at points of $L$ so there's a universal 1-form on $T^*L$ whose value at $(q, \psi \in \mathrm{Hom}(T_qL;\mathbb{R}))$ on a vector $V \in TT^*L$ is $\psi(\pi_*V)$ where $\pi\colon T^*L \to L$. Take its exterior derivative and you get a canonical symplectic form $\omega_{\mathrm{can}}$.

Moreover given an identification of $L$ with $T^2$ there's a canonical trivialisation of $T^*L$: if $q_1$ and $q_2$ are linear coordinates on $L$ then $dq_1$ and $dq_2$ are directions in the fibre of $T^*L$ and we introduce the 'canonically conjugate momenta' $p_i$ which are just the coordinates in these directions. There's a reason symplectic geometry used to be called the canonical formalism. Anyway, this canonical identification $T^2 \times D^2 \to \nu T^2$ is called the Weinstein framing.

In terms of these coordinates $(q_i, p_i)$ we have $\omega_{\mathrm{can}} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. We will consider a small $\epsilon$ such that the square $[-\epsilon,\epsilon]^2 \subset D^2$ and let $U_\epsilon = T^2 \times [-\epsilon,\epsilon]^2$. Using these coordinates we will define a symplectomorphism of $U_\epsilon \setminus U_{\epsilon/2}$ with itself which we will use to reglue $\nu L$ and perform surgery.

Let $\chi\colon [-\epsilon,\epsilon] \to [0,1]$ be a smooth cutoff function equal to 0 on $[-\epsilon,-\epsilon/3]$ and to 1 on $[\epsilon/3,\epsilon]$. Note that $\epsilon/3 < \epsilon/2$. Define

$$\phi_k\colon U_\epsilon \setminus U_{\epsilon/2} \to U_\epsilon \setminus U_{\epsilon/2}$$

by

$$\phi_k(q_1,p_2,q_2,p_2) = \begin{cases} (q_1,p_2,q_2,p_2) & \text{if} p_2 < \epsilon/2 \\ (q_1 + k\chi(p_1), p_2, q_2, p_2) & \text{else} \end{cases}$$

Consider the (square-looking!) torus given by $\partial U_\epsilon \times S^1$ where the $S^1$ is in the $q_1$-direction. This torus is preserved and $\phi_k$ acts as $k$-times a Dehn twist around the curve $\{\star\} \times S^1$. Note that $\phi_k$ preserves the symplectic structure because only $q_1$ is affected and it's affected by a function of $p_1$ so the symplectic structure changes by something proportional to $dp_1 \wedge dp_1$.

**Definition 4.** *The Luttinger surgery on $L \subset X$ is*

$$X_k(L) = (X \setminus U_{\epsilon/2}) \cup_{\phi_k} U_\epsilon$$

*and this is symplectic because $\phi_k$ is symplectic. With a bit more care about the choice of $\chi$ one can ensure that this is independent of all choices, etc. See the paper of Auroux-Donaldson-Katzarkov* [1].

3.2. **Rigidity.** Of course one can do this surgery on a purely topological level for any framed smooth torus with trivial normal bundle in a 4-manifold. We don't need it to be Lagrangian or the framing to be Weinstein. In that case, however, the resulting manifold might not be symplectic. If it's not symplectic then we know that the torus/framing pair is not isotopic to a Lagrangian/Weinstein pair. One can therefore think of two ways to use Luttinger surgery:

- To produce new examples of symplectic manifolds (e.g. people use it these days to produce smooth 4-manifolds homeomorphic but not diffeomorphic to small blow-ups of $\mathbb{CP}^2$). See [3] for a survey.

- To rule out the existence of Lagrangian tori in certain smooth isotopy classes of tori.

It is the latter which we mean by rigidity and which we will now focus on.

**Theorem 5** (Luttinger [5])**.** *There are smoothly embedded tori $T^2 \to \mathbb{R}^4$ which are not isotopic to any Lagrangian torus.*

Note that any smoothly immersed torus is isotopic through immersions to an immersed Lagrangian torus. This is an h-principle, so the phenomenon described by Luttinger's theorem is rightly called a rigidity result.

Explicitly, Luttinger's examples are constructed as follows:

- Take a knot $K$ in the upper-half space in $\mathbb{R}^3$, i.e. in $\mathbb{R}^2 \times [0, \infty)$.
- Identify $\mathbb{R}^4$ minus a neighbourhood of a line with the upper-half space crossed with a circle.
- Take $K$ times the circle. We call this a spun knot because it's obtained by 'spinning' $K$ around the line.

**Theorem 6** (Luttinger)**.** *If $K$ is not the unknot then $K \times S^1 \to \mathbb{R}^4$ is not isotopic to a Lagrangian torus.*

The idea is to assume it were isotopic to a Lagrangian, to perform Luttinger surgery and obtain a manifold $\mathbb{C}^2_k(L)$ which admits no symplectic structure. However, we need to know what framing to use, that is: if $L$ were Lagrangian, what would its Weinstein framing be?

**Proposition 7** (Luttinger [5], Eliashberg-Polterovich [2])**.** . *The Weinstein framing of a Lagrangian torus in $\mathbb{C}^2$ is Seifert. In other words the longitudes are null-homologous in the complement of the torus.*

We'll need one hard theorem to prove this. The theorem is also used later to prove that $\mathbb{C}^2_k(L)$ is not symplectic. It is proved using holomorphic curves.

**Theorem 8** (Gromov [4])**.** *Let $X$ be a symplectic manifold such that*

- $\chi(X) = 0$,
- *there exists a compact subset $C \subset X$ such that $X \setminus C$ is symplectomorphic to the complement of a ball in $\mathbb{C}^2$,*

*then $X$ is symplectomorphic to $\mathbb{C}^2$.*

Note that $\chi(\mathbb{C}^2_k(L))$ by the additivity properties of the Euler characteristic.

**Corollary 9.** *If $L \to \mathbb{C}^2$ is Lagrangian and we do $k$-Luttinger surgery with the Weinstein framing then the result is simply-connected.*

## 4. Proofs

*Proof of Luttinger's theorem, assuming the Proposition.* We'll show that

$$\pi_1(\mathbb{C}^2_k(L)) = 1$$

Note that the meridian of $K$ in the upper-half space is a meridian for $L$, and that the Seifert framing is given by: $\alpha$ equal to the Seifert pushoff of $K$ in the upper-half space and $\beta$ equal to the $S^1$-orbit of a point on $\alpha$. Now the Luttinger surgery reduces to Dehn surgery on each upper-half space and gluing in the line by van Kampen theorem we get $\pi_1(\mathbb{C}^2_k(L)) \cong \mathbb{R}^3_k(K)$. But a theorem of Culler- Gordon-Luecke-Shalen in 3-manifold theory tells us that if $K$ is nontrivial, only finitely many of these surgeries can have trivial fundamental group. □

*Proof of Proposition.* Let $\alpha$ and $\beta$ be a Weinstein framing and do $k$-Luttinger surgery along $\alpha$. The Corollary to Gromov's theorem implies that the Mayer-Vietoris sequence for this new decomposition $\mathbb{C}^2_k(L) \cong \mathbb{C}^2 = (\mathbb{C}^2 \setminus \nu L) \cup \nu L$ is

$$0 \to H_1(T;\mathbb{Z}) \to H_1(\mathbb{C}^2 \setminus L;\mathbb{Z}) \oplus H_1(L;\mathbb{Z}) \to 0$$

Suppose that the Weinstein framing is given by $\alpha = (A,(1,0))$, $\beta = (B,(0,1))$. Then the matrix of this Mayer-Vietoris isomorphism is

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ A & B & 1 \end{pmatrix}$$

But for all $k$ this is an isomorphism of $\mathbb{Z}$-modules and hence its determinant

$$1 + kA$$

is equal to $-1$ for all $k$, hence $A = 0$. Doing Luttinger surgery along $\beta$ gives $B = 0$ similarly. Hence Weinstein pushoffs are Seifert (nullhomologous in the complement)! □

One can think of the proposition itself as a form of rigidity. A Weinstein framing must be Seifert, i.e. among maps $T^*L \to \mathbb{C}^2$ such that the restriction to the zero section is Lagrangian, many are not isotopic to a symplectic embedding!

## 5. Conclusion

We have seen how to do Luttinger surgery on symplectic 4-manifolds. The moral of this talk is that: surgery is not only about constructing new examples! Sometimes you know that the examples are impossible to construct (in our case $\mathbb{C}^2_k(L) \not\cong \mathbb{C}^2$) and this obstruction is translated by the surgery into another obstruction (in our case the knot type of $L$). This seems to be something quite common in symplectic geometry.

## References

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