# Quantum cohomology of twistor spaces

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3rd February 2012

I would like to thank the organisers for the opportunity to speak here, at the sixth Workshop on Symplectic Geometry, Contact Geometry and Interactions. I would also like to thank the organisers of the previous five workshops. This series began the year I started my PhD so I have grown up with this wonderful opportunity to learn and to meet other mathematicians. So thank you, your hard work is much appreciated.

Today I want to introduce you to some unusual symplectic manifolds. They are very easy to describe:

$$Z = \Gamma \backslash SO^+(2n,1)/U(n)$$

as quotients of homogeneous spaces by cocompact torsionfree lattices. Of course finding such a lattice in $SO^+(2n,1)$ is a nontrivial problem we will not concern ourselves with, but rest assured that they exist (there are infinitely many arithmetic examples).

The structure of these spaces becomes more apparent if we notice that we can quotient out on the right by a bigger group $SO(2n) \supset U(n)$:

$$\tau \colon Z = \Gamma \backslash SO^{+}(2n,1)/U(n) \to \Gamma \backslash SO^{+}(2n,1)/SO(2n) = M$$

The fibre of this projection is just the quotient $F = SO(2n)/U(n)$. On the right-hand side we have a compact quotient of $SO^{+}(2n,1)/SO(2n)$ but that's just hyperbolic space $\mathbb{H}^{2n}$, so the target of this projection is a hyperbolic $2n$-manifold. The total space $Z$ is therefore an $SO(2n)/U(n)$-bundle over the hyperbolic manifold $M$.

We can see immediately from the homotopy long exact sequence of this fibration that

$$\Gamma = \pi_1(M) \cong \pi_1(Z)$$

so that the spaces $Z$ have the fundamental group of a hyperbolic $2n$-manifold. Using the theory of harmonic maps one can prove:

**Theorem (Carlson-Toledo)**

*No compact Kähler manifold can have the fundamental group of a hyperbolic $m$-manifold if $m > 2$.*

**Corollary**

*The manifolds $Z$ are non-Kähler (when $n > 1$).*

You can easily calculate the dimension of $Z$:

$$\dim(Z) = \dim(F) + \dim(M) = n(n-1) + 2n = n(n+1)$$

Moreover we can understand the fibre thanks to the accidental isomorphisms in low-dimensions:

$$SO(2)/U(1) = \{\star\}$$
$$SO(4)/U(2) = \mathbb{CP}^1$$
$$SO(6)/U(3) = \mathbb{CP}^3$$
$$SO(8)/U(4) = \text{quadric 6-fold}$$

The spaces $Z$ are a special case of a more general construction: the twistor bundle. I'll explain this in its full generality, then return to hyperbolic manifolds later.

### Definition

*Given a Riemannian* $(M, g)$ *the twistor bundle* $Z$ *is just the bundle* $\tau\colon Z \to M$ *whose fibre at* $p$ *is the space of* $g_p$*-orthogonal complex structures on* $T_pM$*. A complex structure* $\psi$ *on a Euclidean vector space is orthogonal if*

$$g_p(\psi V, \psi W) = g_p(V, W).$$

*Don't get confused.* $M$ *doesn't have to be almost complex: an orthogonal almost complex structure on* $M$*, if it exists, is a section of the twistor bundle.*

I'll now define some geometric structures on the twistor space of a general Riemannian manifold.

$Z$ sits inside the bundle $\mathrm{End}(TM)$ and it is preserved by the Levi-Civita connection. Therefore $TZ$ gets a horizontal-vertical decomposition

$$T_\psi Z = \mathcal{H}_\psi \oplus \mathcal{V}_\psi$$

with $\mathcal{H} \cong \tau^* TM$ and $\mathcal{V}_\psi = T_\psi F$, $F$ being the fibre through $\psi$.

- Since $F$ is a homogeneous space it has a natural $SO(2n)/U(n)$-invariant Kähler structure $(\omega_F, j_F, g_F)$.
- $\mathcal{H}$ has a tautological almost complex structure: a point $\psi$ in $Z$ defines a complex structure $\psi$ on $T_pM$ (where $p = \tau(\psi)$) and hence on the horizontal space $\mathcal{H}_\psi \overset{\tau_*}{\cong} T_pM$. We call this tautological a.c.s $\psi$. We can also pull back the metric $g$ to $\mathcal{H}$ and since $g$ and $\psi$ are orthogonal they define for us a compatible 2-form $\omega_\psi$ on $\mathcal{H}$.

Putting all this together, using the splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ we have:

- a metric $g = \tau^* g \oplus g_F$,
- two almost complex structures $J_\pm = \psi \oplus (\pm j_F)$,
- two nondegenerate 2-forms $\omega_\pm = \omega_\psi \oplus (\pm\omega_F)$ compatible with $J_\pm$ respectively.

Despite the apparent simplicity of the definition, the 2-forms $\omega_\pm$ are usually not closed. This is because the distribution $\mathcal{H}$ is usually not integrable.

Similarly, $J_\pm$ are usually not integrable.

**Proposition**

*$J_+$ is integrable if and only if $(M, g)$ is either 4-d and self-dual or higher-dimensional and conformally flat. $J_-$ is never integrable.*

Reznikov (1993) defined another 2-form $\omega_{\mathrm{rez}}$ which is always closed, but not always nondegenerate...

Recall that the Riemann curvature tensor can be thought of as a map

$$\hat{R}\colon \Lambda^2 TM \to \Lambda^2 TM$$

(this is just a rephrasing of the antisymmetries of $R_{ijkl}$). Reznikov's form is

$$\omega_{\mathrm{rez}} = -\hat{R}(\omega_\psi) \oplus (-\omega_F)$$

Note that when $M$ is hyperbolic $\hat{R} = -1$ so

**Proposition**

*When $(M, g)$ is hyperbolic, $\omega_{\mathrm{rez}} = \omega_-$ is a $J_-$-compatible symplectic form.*

**Theorem (Reznikov (1993), Fine-Panov (2009))**

- *(Reznikov) When g is sufficiently negatively curved (e.g. pinched curvature)* $\omega_{\mathrm{rez}}$ *is symplectic.*
- *(Fine-Panov) When g is hyperbolic,* $c_1(Z, J_-) = (n-2)[\omega_{\mathrm{rez}}]$*, so Z is symplectic Calabi-Yau when* $n = 2$ *and symplectic Fano when* $n \geq 3$.

Henceforth we restrict attention to the twistor spaces of hyperbolic manifolds, which have the homogeneous space description I began with

$$\Gamma \backslash SO^+(2n, 1)/U(n)$$

I will also assume $n = 3$ for simplicity, so these are $\mathbb{CP}^3$-bundles over hyperbolic 6-manifolds.

These are extremely unusual symplectic Fano 12-manifolds. Let's contrast them with complex projective Fano varieties

- Fano varieties are simply-connected. The spaces $Z$ have hyperbolic fundamental groups.
- In each dimension there are only finitely many diffeomorphism types of Fano variety. There are infinitely many arithmetic lattices one can pick for $\Gamma$ t give infinitely many non-diffeomorphic $Z$.
- Fano varieties have lots of holomorphic curves. In particular, given any two points one can find a stable holomorphic genus 0 curve passing through them (they are *rationally connected*). By contrast, we will see that, in $Z$, given points in different fibres $F$ there are no $J_-$-holomorphic curves connecting them.

This is all the more surprising when we remember that Taubes/Li-Liu proved that any symplectic Fano 4-manifold is deformation equivalent to a Fano surface. The same is clearly not true in dimension 12.

However, there are some properties these spaces share in common with Fano varieties.

**Theorem (E. '11)**

*Let $Z$ be the twistor space of a compact orientable hyperbolic 6-manifold.*

- *$Z$ is uniruled, i.e. for any $\omega_-$-compatible a.c.s. $J$ there is a $J$-holomorphic genus 0 curve through every point.*
- *The quantum cohomology of $Z$ contains an invertible element in degree 2.*

*Indeed the quantum cohomology ring is*

$$QH(Z;\mathbb{C}(q)) = H^*(M;\mathbb{C}[q])[\alpha]/(\alpha^4 = 8\alpha\tau^*\chi + 8q\alpha^2 - 16q^2)$$

*where $\alpha = c_1(Z)$ and $\chi$ is the Euler class of $TM$.*

The existence of an invertible element in degree 2 certainly feels very Fano-like (like the hyperplane in $\mathbb{CP}^n$). One can deduce from this corollaries such as: any aspherical Lagrangian in $Z$ has minimal Maslov number 2 (using Damian's proof of the Audin conjecture).

I want to prove uniruledness. To do this, we need to get a feel for the $J_-$-holomorphic curves in $Z$.

**Theorem (Eells-Salamon (1985) twistor correspondence)**

*Let $Z$ be the twistor space of ANY Riemannian $2n$-manifold. A $J_-$-holomorphic curve $u\colon \Sigma \to Z$ projects to either a point or a branched minimal immersion of $\Sigma$ in $M$ and any branched minimal immersion has a $J_-$-holomorphic curve living over it.*

Now when $M$ is hyperbolic, $\pi_2(M) = 0$ and so any map $S^2 \to M$ is nullhomotopic. Now a branched minimal immersion is a harmonic map, but the harmonic map energy is convex on the space of maps into a negatively curved manifold (that's why there's a unique closed geodesic in each homotopy class on a hyperbolic manifold) so there is an essentially unique harmonic map of the sphere into $M$, namely the constant map!

**Corollary**

*All $J_-$-holomorphic curves in the twistor space of a hyperbolic manifold live in the fibres of the twistor fibration.*

We see at once that the twistor space of a hyperbolic manifold is not rationally connected. However, we do see a $J_-$-holomorphic sphere through every point: every point is contained in some fibre which is just $\mathbb{CP}^3$ with its standard complex structure. To prove uniruledness we need to show that there is a non-vanishing Gromov-Witten invariant counting curves passing through a fixed point, i.e. that the evaluation map

$$\mathrm{ev}\colon \mathcal{M}_{0,1}(\beta, J) \to Z$$

(for genus 0 curves in some class $\beta \in H_2(Z;\mathbb{Z})$) has nonzero degree for some generic $J$. The problem is that $J_-$ is not generic. The moduli spaces of $J_-$-holomorphic curves have the wrong dimension. Let's see this for lines in $\mathbb{CP}^3$ (twistor lines). The expected dimension for the moduli space $\mathcal{M}_{0,1}(\beta, J)$ is

$$2n + 2c_1(\beta) - 6 + 2$$

We have $c_1(\beta) = 2$ and $n = 6$ so we get the expected dimension being 12. However there is a $\mathbb{CP}^2$ of lines through every point (and there are 12 dimensions of points) so the moduli space $\mathcal{M}_{0,1}(\beta, J_-)$ has dimension 16.

However, the moduli space is clearly compact (these are minimal area spheres) and smooth. Moreover you can check that its tangent spaces are equal to the kernels of the linearised $\overline{\partial}_{J_-}$-operator. That is to say: over a suitable Banach space $\mathcal{B}$ of maps $u\colon S^2 \to Z$ representing the homology class $\beta$ there is a section $\overline{\partial}_{J_-}(u)$ of the Banach bundle $\Omega^{0,1}_{J_-}(S^2, u^*TZ)$ (measuring the failure of $u$ to be $J_-$-holomorphic) and that section vanishes *cleanly*[1], rather than transversely. Checking this requires solving a linear elliptic PDE and amounts to nothing more than an application of the maximum principle.

## Proposition (McDuff-Salamon, Proposition 7.2.3)

*When $\overline{\partial}_J$ vanishes cleanly and the moduli space is compact the Gromov-Witten class is*

$$\mathrm{ev}_*\mathrm{PD}(\mathrm{eul}(\mathfrak{Obs}))$$

*where $\mathfrak{Obs}$ is the* obstruction bundle, *a vector bundle built out of the cokernels of the linearised $\overline{\partial}_J$-operators.*

[1] i.e. the section intersects the zero-section cleanly, i.e. the tangent spaces of the intersection are the intersections of the tangent spaces.

Again we can easily solve for the kernel of the adjoint of the linearised $\overline{\partial}_J$-operator (which is the fibre of the obstruction bundle). It turns out that given a vector $V \in T_pM$ one can construct an element $\sigma_V \in \mathfrak{Obs}_u$ for any curve $u$ whose image is contained in the twistor fibre at $p$. The element of $\mathfrak{Obs}_u$ only depends on the projection of $V$ to some 4-dimensional subspace, since we know that $\mathfrak{Obs}_u$ is a rank 4 vector bundle (the expected and actual dimensions differ by 4). To understand what this vector bundle is, we give a different description of the moduli space of twistor lines.

Recall that $SO(4)/U(2) \cong \mathbb{CP}^1$. Note that this is diffeomorphic to $SO(4) \times SO(2)/U(2) \times U(1)$, which parametrises orthogonal complex structures on a 6-dimensional vector space for which some fixed 4-plane is complex, hence it is a subset of $SO(6)/U(3)$, the twistor fibre. Indeed, it is precisely a complex line in $\mathbb{CP}^3$. The moduli space of complex lines in $\mathbb{CP}^3$ is therefore identified with the space of 4-planes in $\mathbb{R}^6$ and, reassuringly there are $U(3)/U(2) \times U(1) = \mathbb{CP}^2$ of them through every point. Moreover, the element $\sigma_V \in \mathfrak{Obs}_u$ only depends on the projection of $V$ to the 4-plane corresponding to $u$. This implies:

## Proposition

*The moduli space of twistor lines in* $Z$ *is*

$$\mathcal{M}_{0,0}(\beta, J_-) = \Gamma \backslash SO^+(6,1)/SO(4) \times SO(2)$$

*or*

$$\mathcal{M}_{0,1}(\beta, J_-) = \Gamma \backslash SO^+(6,1)/U(2) \times U(1)$$

*where the evaluation map to* $\Gamma \backslash SO^+(6,1)/U(3)$ *is precisely the map induced by the inclusion* $U(2) \times U(1) \to U(3)$. *Moreover the obstruction bundle is the canonical* $SO(4)$*-bundle (resp.* $U(2)$*-bundle).*

The theorem on obstruction bundles now tells us to fibre-integrate the Euler class of this $U(2)$-bundle (i.e. its second Chern class) along the evaluation map. This corresponds to integrating

$$\mathrm{ev}_! c_2 = [Z] \cdot \int_{\mathbb{CP}^2} c_2 = [Z]$$

in each fibre and the answer is just $[Z]$, i.e. for a generic $J$ the evaluation map has degree 1, proving uniruledness.

Higher degree curves may contribute to the 3-point Gromov-Witten invariants and for these the moduli spaces are not compact. Nonetheless, it turns out that for all the 3-point calculations involved in the quantum product one can, using a well-chosen vector field on $M$, construct a nonvanishing element of $\mathfrak{Obs}_u$ for all $u$ in all strata of the moduli space. If one employs the theory of Kuranishi structures, this is certainly enough to ensure the contribution vanishes, but given we are in a monotone setting I wanted to appeal only to the standard (Ruan-Tian/McDuff-Salamon) theory of curves in monotone manifolds. One must carefully prove a version of the obstruction bundle theory (using the implicit function theorem and gluing) in this setting and the result is that the higher degree curves do not contribute to the 3-point Gromov-Witten invariants.

I want to finish by introducing some beautiful monotone Lagrangian submanifolds $L_\Sigma \subset Z$. These were discovered by Reznikov (1993).

**Definition (Reznikov Lagrangians)**

*Let $\Sigma \subset M$ be a totally geodesic submanifold of dimension $n$ (half the dimension of $M$). The subset $L_\Sigma \subset \tau^{-1}(\Sigma)$ consisting of complex structures $\psi$ such that $\psi(T\Sigma) \perp T\Sigma$ is a Lagrangian submanifold of $Z$, diffeomorphic to the frame bundle of $\Sigma$. When $n \geq 3$ these are monotone Lagrangians.*

A similar argument with the Eells-Salamon twistor correspondence shows that all holomorphic discs with boundary on $L_\Sigma$ are vertical (contained in twistor fibres) and in the case $n = 3$, $L_\Sigma \cap F = SO(3) = \mathbb{RP}^3 \subset \mathbb{CP}^3$. Analogous calculations to those with the holomorphic spheres imply that there are two Maslov 2 holomorphic discs through every point (they're Lagrangian uniruled) and that the self-Floer cohomology is $H^*(L)$ (with suitable Novikov coefficients).

There is a folklore theorem inspired by mirror symmetry (proved in the monotone toric case by Auroux) that the number $\mu_0$ which counts Maslov 2 discs (suitably weighted by areas and holonomies of a local system) through a point of a Lagrangian with non-vanishing self-Floer cohomology also arises as an eigenvalue for the action of $c_1(Z)$ on $QH(Z)$ by quantum product. If one uses the presentation of the quantum cohomology ring I gave earlier then one sees that these eigenvalues are $\pm 2\sqrt{q}$ (which indeed arise as the counts of discs on Reznikov Lagrangians equipped with different spin structures) and the roots of the equation

$$\lambda^4 - 8q\lambda^2 - 8\chi(M)\lambda + 16q^2$$

It would be intriguing to find (or rule out) Lagrangians corresponding to these eigenvalues.

For example, when $M = S^6$ the twistor space is $SO(7)/U(3) \to SO(7)/SO(6)$ but $SO(7)/U(3)$ is a quadric 6-fold (it's the same as $SO(8)/U(4)$!). There are plenty of Reznikov Lagrangians (e.g. over equatorial $S^3$s) but there is also a Lagrangian 6-sphere which predicts diffeomorphically to the base (I think this is just $G_2/SU(3)$, but I haven't checked). Our conjectural Lagrangians have to see the whole base somehow (its Euler class pops up in their disc counts) but they cannot be sections as the base is hyperbolic and the total space is uniruled (this would contravene a theorem of Viterbo/Eliashberg).