Lagrangian knottedness and unknottedness in rational surfaces

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Outline:

- Lagrangian knottedness
  - Symplectic geometry of complex projective varieties,
  - $\mathbb{D}_5$,
  - Lagrangian spheres and Dehn twists
- Lagrangian unknottedness
  - Hind's theorem on $S^2 \times S^2$,
  - $\mathbb{D}_2$, $\mathbb{D}_3$, $\mathbb{D}_4$,
  - ...

Symplectic geometry

A symplectic manifold is a pair

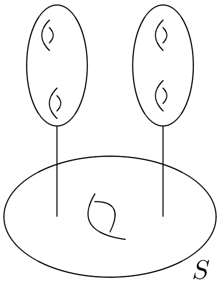
$$(\underset{\substack{\nwarrow \\ 2n - \text{dimensional manifold}}}{X}, \underset{\substack{\nearrow \\ \text{closed, non-degenerate 2-form}}}{\omega})$$

A symplectic manifold admits an infinite-dimensional group of diffeomorphisms preserving ω (*symplectomorphisms*) denoted by $\mathrm{Symp}(X)$. Examples include:

- The phase space $\{(x_1, \ldots, x_3, p_1, \ldots, p_3) \ :\}$ of a classical dynamical system. The symplectic form is $\sum_{i=1}^{3} dx_i \wedge dp_i$ and the time evolution associated to a Hamiltonian function is a symplectomorphism. Symplectic geometry is a good setting for thinking about problems with periodic orbits of Hamiltonian flows.
- Any complex projective variety inherits a Kähler form from the ambient projective space. This is in particular a symplectic form. **We know very little about the symplectic geometry of complex projective varieties.** We will concentrate on complex surfaces (4-dimensional symplectic manifolds). Even for these little is known, except in the case of rational surfaces.

The next task is to see what symplectic geometry brings to the study of projective varieties.

Consider a family of smooth projective varieties (with fixed Kähler class) over some base S. The study of such families is the core of moduli theory. We will see that symplectic geometry offers a new perspective.



Basic idea from Hodge theory: One can replace the fibres of this family with the homology groups of the fibres to obtain a vector bundle over S and there is a natural connection on this vector bundle called the Gauss-Manin connection. The monodromy of this connection gives a representation

$$\pi_1(S) \to \mathrm{Aut}(H_*(X, \mathbb{C}))$$

Symplectic monodromy: In fact we can do better by introducing a symplectic connection on the family itself. The monodromy around a loop in S is now a symplectomorphism ϕ of the fibre. Varying the loop or the connection changes ϕ but the monodromy is always a symplectomorphism. Therefore we get a distinguished path component of $\pi_0(\mathrm{Symp}(X))$ and we have enhanced our monodromy representation

$$\pi_1(S) \to \pi_0(Symp(X))$$

This contains strictly more information than the homological counterpart mentioned earlier. In fact, we also get a map

$$S \to B\mathrm{Symp}(X)$$

well-defined up to homotopy.

$\mathbb{D}_5$

It's high time for an example. Consider $\mathbb{D}_5$, the blow-up of the projective plane at 5 points in general position. For simplicity let's order the points. We have a universal family of blow-ups over the configuration space

$$S = \mathrm{Conf}_5^{\mathrm{gen,ord}}(\mathbb{CP}^2)/\mathbb{P}GL(3, \mathbb{C})$$

There is a unique smooth conic $\mathbb{CP}^1$ through five points in general position. Therefore to every configuration in S we can assign a configuration of five points in $\mathbb{CP}^1$:

$$S \to \mathrm{Conf}_5^{\mathrm{ord}}(\mathbb{CP}^1)/\mathbb{P}GL(2, \mathbb{C})$$

and in fact this assignment is a homeomorphism. The configuration space on the right is a classifying space for the pure braid group on 5 strands on S^2.

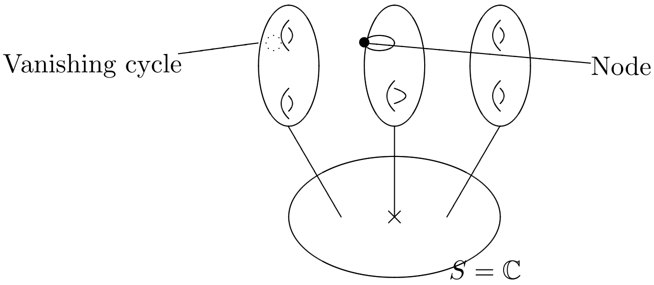
Theorem 1 (Seidel)**.** *The map $S \to BSymp(\mathbb{D}_5)$ induces injections on all homotopy groups. In particular, the component group of $Symp(\mathbb{D}_5)$ contains the pure braid group on 5 strands on S^2.*

This uses *pseudoholomorphic curves.* The idea is that there's still a smooth rational curve through the five exceptional curves in $\mathbb{D}_5$ when you work with non-integrable almost complex structures.

Lagrangian spheres and Dehn twists

I now introduce some ideas which help us understand the symplectomorphisms in the example above more geometrically. First of all, a **Lagrangian submanifold** is an n-dimensional submanifold of X on which ω vanishes. These arise and are important in all the examples of symplectic manifolds mentioned before, but we'll concentrate on those relevant for projective varietes.

Vanishing cycles: Suppose we have a family of varieties over $\mathbb{C}$ which are all smooth except the fibre over 0 which has a single nodal singularity. Pick a path from x to 0 and look at the symplectic monodromy along this path. Some set of points gets crushed to the node: in the illustration it's a circle. In fact, this set is always a Lagrangian sphere L. Varying the path changes L but only through *Lagrangian isotopies.*



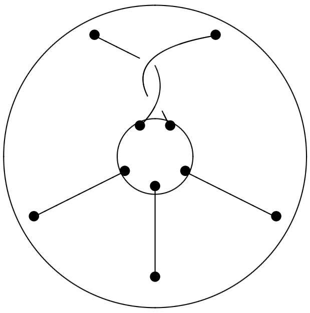
Monodromy: The monodromy around the unit circle is called a **Dehn twist** in L. In a family of curves this is precisely the usual Dehn twist in a circle.

Example: Dehn twists for $\mathbb{D}_5$.

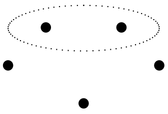
Suppose we bring two of the ordered points in $\mathbb{CP}^2$ together along a complex line joining them. This gives a degeneration as above but in fact it's not quite nodal: the monodromy is the *square* of a Dehn twist in a Lagrangian sphere. This is because we ordered the points: it's possible to get rid of this by base-change.

Under Seidel's map from the pure braid group to the group of path components of the symplectomorphism group, this squared Dehn twist came from the braid which is a full twist around a particular pair of points (see A).

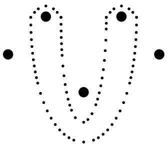
A)



B)



C)



We can represent this by the circle in B. Suppose instead we performed the full twist represented by the circle in C. This corresponds to a Lagrangian sphere and since it is a different pure braid Seidel's theorem shows that the Dehn twist is not isotopic to the one given by B. If two Lagrangian spheres are isotopic through Lagrangian spheres then the corresponding Dehn twists are isotopic through symplectomorphisms. Hence the Lagrangian spheres corresponding to B and C are not isotopic through Lagrangian spheres. It's not too hard to show that they are isotopic as smooth submanifolds.

This phenomenon we call **Lagrangian knotting**.

Lagrangian unknottedness

Seidel's work has shown that knottedness is the rule rather than the exception. Let's study some exceptions.

Theorem 2 (Hind)**.** *There is a unique Lagrangian sphere up to isotopy in the quadric surface* $S^2 \times S^2$ *with* $\omega = \omega_0 \oplus \omega_0$

In $S^2 \times S^2$ there is a well-known Lagrangian sphere, the *antidiagonal*:

$$\{(x, -x) : x \in S^2\}$$

where $-$ denotes the antipodal map. Hind's theorem says that every Lagrangian sphere in $S^2 \times S^2$ is isotopic to that one.

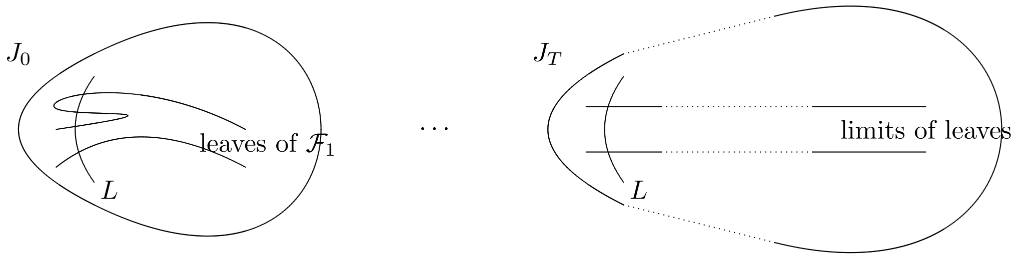
Theorem 3 (Hind)**.** *There is a unique Lagrangian sphere up to isotopy in the cotangent bundle* T^*S^2*.*

T^*S^2 is the phase space of classical dynamics on S^2. The locus where all conjugate momenta vanish is Lagrangian (the *zero section*). Hind's theorem says that every Lagrangian sphere in T^*S^2 is isotopic to that one.

Idea behind proof for $S^2 \times S^2$: Given a Lagrangian sphere L,

1. Construct a pair of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $S^2 \times S^2$ whose leaves are pseudoholomorphic curves in the homology classes $S^2 \times \{\star\}$ and $\{\star\} \times S^2$, such that L intersects each leaf exactly once, transversely (this certainly exists for the antidiagonal sphere).
2. Use these foliations to construct an isotopy between L and the antidiagonal sphere.

The second part is a relativly easy application of Moser's argument. The hard part is part 1, which uses symplectic field theory (SFT).



One splits $S^2 \times S^2$ into W and V where W is a neighbourhood of L and V is its complement. Write M for the interface and N for a collar neighbourhood of M. Then one constructs a family of almost Kähler metrics g_t which make N into a longer and longer neck. For each of these, Gromov's theory of pseudoholomorphic curves gives us the pseudoholomorphic foliations we desire and all that remains is to show that for large t the leaves intersect L once transversely. To see this, take the limit as $t \to \infty$: the leaves break up into punctured holomorphic parts in V and W which we can analyse using methods of SFT. The result is that the W-part of a leaf must be a single holomorphic plane intersecting L once transversely. This must also be true of leaves for large t by the nature of the convergence.

$\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$

Complex curves can be classified coarsely by looking at the canonical class. There is a unique curve C with $c_1(TC) > 0$, namely $\mathbb{CP}^1$. The Enriques-Kodaira classification of complex surfaces tells us that there are ten surfaces S with $c_1(TS) > 0$, namely:

$$\mathbb{CP}^2 \quad \mathbb{D}_1 \quad \mathbb{D}_2 \quad \mathbb{D}_3 \quad \mathbb{D}_4 \quad \mathbb{D}_5 \quad \mathbb{D}_6 \quad \mathbb{D}_7 \quad \mathbb{D}_8$$

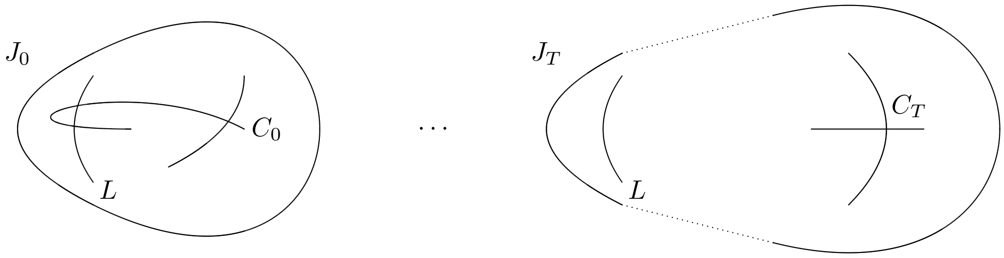
$$S^2 \times S^2$$

Neither $\mathbb{CP}^2$ nor $\mathbb{D}_1$ contain any Lagrangian spheres for homological reasons. $\mathbb{D}_5$ and higher blow-ups exhibit Lagrangian knotting phenomena. $S^2 \times S^2$ contains a unique isotopy class of Lagrangian spheres. What about $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$? If we try and construct a universal family over configurations of points in $\mathbb{CP}^2$, we get no interesting phenomena because $\mathbb{P}GL(3, \mathbb{C})$ acts transitively on quadruples of points in $\mathbb{CP}^2$. This suggests the following

Theorem 4 (E.). *If L and L' are homologous Lagrangian spheres in one of $\mathbb{D}_2$, $\mathbb{D}_3$ or $\mathbb{D}_4$ then they are isotopic through Lagrangian spheres.*

Idea of proof for $\mathbb{D}_2$: $\mathbb{D}_2$ is the blow-up of $S^2 \times S^2$ at a single point. If we can disjoin L from the exceptional curve of this blow-up and from the proper transform of the diagonal then we have effectively moved it into a neighbourhood of the antidiagonal sphere. This neighbourhood is isomorphic to a neighbourhood in T^*S^2, but Hind proved isotopy uniqueness for Lagrangian spheres in T^*S^2. Therefore the proof proceeds as follows:

1. Find a family J_t of almost complex structures and a configuration C_t of J_t-holomorphic spheres representing the exceptional sphere and the proper transform of the diagonal such that J_0 and C_0 are standard and such that C_T is disjoint from L.
2. Construct a global isotopy $\phi_t : \mathbb{D}_2 \to \mathbb{D}_2$ such that $C_t = \phi_t(C_0)$ and set $L_t = \phi_t^{-1}(L)$ to get an isotopy of L which disjoins it from C_0.
3. Now appeal to Hind's theorem to see that L is unique up to isotopy since the complement of C_0 contains a unique Lagrangian sphere up to isotopy.



This leaves wide open the classification of Lagrangian spheres in $\mathbb{D}_5$. However, the following theorem is suggestive:

Theorem 5 (E.). *The map $S \to BSymp_0(\mathbb{D}_5)$ defined earlier (where S is the classifying space of the pure braid group on five strands for S^2 and $Symp_0$ denotes symplectomorphisms acting trivially on homology) is a homotopy equivalence.*

This implies that if there were any new "exotic" Lagrangian spheres other than the ones we already constructed as vanishing cycles, the corresponding Dehn twists would be indistinguishable from those around the vanishing cycles. This leads me to

Conjecture 1. *The only Lagrangian spheres in $\mathbb{D}_5$ are those arising as vanishing cycles from nodal algebraic degenerations.*